

# Taylor Rules, McCallum Rules and the Term Structure of Interest Rates

Michael F. Gallmeyer<sup>1</sup>

Burton Hollifield<sup>2</sup>

Stanley E. Zin<sup>3</sup>

November 2004

*Prepared for the Carnegie-Rochester Conference (Preliminary and Incomplete)*

JEL classification codes: G0, G1, E4

Keywords: term structure, monetary policy, Taylor rule.

---

<sup>1</sup>The David A. Tepper School of Business, Carnegie Mellon University.

<sup>2</sup>The David A. Tepper School of Business, Carnegie Mellon University.

<sup>3</sup>The David A. Tepper School of Business, Carnegie Mellon University, and NBER

## Abstract

Recent empirical research shows that a reasonable characterization of federal-funds rate targeting behavior is that the change in the target rate depends on the maturity structure of interest rates and exhibits little dependence on lagged target rates. See, for example, Cochrane and Piazzesi (2002). The result echoes the policy rule used by McCallum (1994) to rationalize the empirical failure of the ‘expectations hypothesis’ applied to the term-structure of interest rates. McCallum showed that a desire by the monetary authority to adjust short rates in response to exogenous shocks to term premiums imbedded in long rates along with a desire for interest smoothing can generate term structures that account for the puzzling regression results of Fama and Bliss (1987). That is, rather than forward rates acting as unbiased predictors of future short rates (or equivalently, current forward premiums acting as unbiased predictors of the future short-rate changes), the historical evidence suggests that the correlation between forward rates and future short rates is surprisingly low. McCallum also clearly pointed out that this reduced-form approach to the policy rule, although naturally forward looking, needed to be studied further in the context of more realistic response functions such as the now standard Taylor (1993) rule. We model the term premium endogenously and solve for equilibrium term structures in environments in which interest-rate targeting follows a Taylor rule. Our first goal is to demonstrate that the simultaneous determination of interest-rate targets and long rates results in endogenous equilibrium term premiums in which the Taylor rule and the ‘McCallum rule’ are observationally equivalent. That is, as a positive statement, the Taylor rule is well-approximated in equilibrium by the forward-looking behavior of the McCallum rule. This arbitrage-free approach, similar in spirit to Ang and Piazzesi (1993), helps us reconcile observed policy behavior with observed term structure behavior. Our second goal is to ask a more normative question about the interest-rate smoothing behavior of the monetary authority. In particular, in what sense are these rules optimal from the perspective of private agents in the economy? We show that when such a rule can offset the effects of a risk premium on agents optimality conditions, the McCallum Rule forms an optimal monetary policy.

# 1 Introduction

Understanding a monetary authority's policy rule is a central question of monetary economics. Understanding the determinants of the term structure of interest rates is a central question of financial economics. Combining the two creates an important link across these two related areas of economics, and has been the focus of a growing body of theoretical and empirical research. Since the work by Mankiw and Miron (1986) which established a clear change in the dynamic behavior of the term structure after the founding of the Federal Reserve, researchers have been working to uncover precisely the relationship between the objectives of the monetary authority and how it feeds back through aggregate economic activity and the objectives of bond market participants, to determine an equilibrium yield curve that embodies monetary policy considerations.

Of particular interest in this area is the work by McCallum (1994a). McCallum showed that by augmenting the expectations-hypothesis model of the term structure with a monetary policy rule that uses an interest rate instrument and that is sensitive to the slope of the yield curve, *i.e.*, the risk premium on long-term bonds, the resulting equilibrium interest rate process is better able to capture puzzling empirical results based on the expectations hypothesis alone. Kugler (1997) established that McCallum's findings extend across a variety of choices for the maturity of the "long bond," as well as across a variety of countries. McCallum (1994b) applied a similar argument to foreign exchange puzzles. Although the policy rule used by McCallum was not an innovation *per se* since it has a relatively long tradition in the literature documenting Fed behavior (see, for example, the descriptions in Goodfriend (1991) and (1993)), McCallum's innovative use of such a rule for empirical term structure analysis leads us to refer to such a yield-curve-sensitive interest-rate-policy rule as a "McCallum Rule." This stands in contrast to other interest-rate policy rules based on macroeconomic fundamentals, such as the well-known "Taylor Rule."

The benefits of integrating macroeconomic models with arbitrage-free term structure

models will depend on the perspective one takes. From a purely empirical asset-pricing perspective, building term-structure models based on macroeconomic factors has proven to be quite successful. Ang and Piazzesi (2003), following work by Piazzesi (2001), have shown that a factor model of the term structure that also imposes no-arbitrage conditions, can provide a better empirical model of the term structure than a model based on unobserved factors or latent variables alone. Estrella and Mishkin (1997), Evans and Marshall (1998) and (2001), also provide evidence of the benefits building arbitrage-free term-structure models with macroeconomic fundamentals.

From a monetary economics perspective, using information about expectations of future macroeconomic variables as summarized in the current yield curve, is attractive since high-quality financial-market data is typically available in real time. Monetary policy rules based this information, therefore, may be better suited for dealing with immediate economic conditions, rather than rules based on more-slowly-gathered macroeconomic data (see, for example, Rudebush (1998), Cochrane and Piazzesi (2002), and Piazzesi and Swanson (2004), among others).

There has been some recent work that seeks to combine these two dimensions. Rudebush and Wu (2004) and Ang, Dong and Piazzesi (2004) investigate the empirical consequences of imposing an optimal Taylor Rule on the performance of arbitrage-free term structure models.

The optimality of McCallum rules, however, has not yet been studied in the same rigorous fashion as the optimality of Taylor Rules. In this paper, we extend the theoretical term-structure models on which these empirical macro-finance studies are based, to include both a formal macroeconomic model and an explicit monetary policy rule. Our goal is summarized by Cochrane and Piazzesi (2002) who estimate a Fed policy rule and find that when setting target rates, “the Fed responds to long-term interest rates, perhaps embodying inflation expectations, and to the slope of the term structure, which forecasts real activity.” In other words, we ask whether a rule that directly responds to macroeconomic fundamentals

such as inflation or inflation expectations and real output or expected real output, *e.g.*, a Taylor Rule, can be equivalent to a McCallum Rule in which short-term interest rates are set in response to term structure considerations, as suggested by the work of Cochrane and Piazzesi. If so, then what are the theoretical restrictions on both the asset-pricing behavior of the economy and macroeconomic behavior of the economy that result in this equivalence? In addition, we find the conditions under which a McCallum Rule is “optimal” from the perspective of private agents in the economy. This parallels the optimality of the Taylor Rule studied in Clarida, Galí and Gertler (1999).

We begin in Section 2 by reviewing McCallum’s basic argument and consider an extension to a broad class of equilibrium term structure models with a particularly convenient linear-factor structure. We extend McCallum’s result to the case of an endogenous risk premium and show that his conclusions are essentially unchanged. In Section 3 we develop a simple “New Keynesian” macroeconomic model based on Clarida, Galí and Gertler (1999) and study the equilibrium in this economy when the monetary authority follows a Taylor Rule. In addition, we establish the conditions under which the McCallum Rule and the Taylor Rule are equivalent. This equivalence depends critically on the link between fundamental macroeconomic shocks to inflation and output and the risk premiums earned in the long-term bond market. This equivalence depends critically on the link between fundamental macroeconomic shocks to inflation and output and the risk premiums earned in the long-term bond market. These findings also suggest how our model can be modified to accommodate other equilibrium term-structure models. In particular, the risk premiums in our basic model are the result of stochastic volatility in the underlying state variables. But recent empirical research (see Dai and Singleton (2002) and (2003)) shows that models where risk premiums result from constant volatility and a “price of risk” that depends on the underlying state variables have much better empirical properties. In Section 4 we show that with plausible parameters for the macroeconomic model and a plausible monetary policy rule, the McCallum Rule is consistent with empirical evidence. The final section summarizes and concludes the paper.

## 2 McCallum meets Duffie-Kan

We begin with a brief review of the McCallum (1994) model of the term structure that embodies an active monetary policy with an interest rate instrument. Denote the price at date  $t$  of a default-free pure-discount bond that pays 1 with certainty at date  $t+n$  as  $b_t^n$ . The continuously compounded yield on this bond,  $r_t^n$ , is defined as:

$$b_t^n \equiv \exp\{-nr_t^n\}, \quad (1)$$

or

$$r_t^n = -\frac{1}{n} \log b_t^n. \quad (2)$$

We refer to the interest rate, or short rate, as the yield on the bond with the shortest maturity under consideration,  $r_t \equiv r_t^1$ . The one-period forward rate,  $f_t^n$ , implicit in the price of an  $n+1$  period bond is defined in a similar way,

$$b_t^{n+1} \equiv b_t^n \exp\{-f_t^n\}, \quad (3)$$

or

$$f_t^n \equiv \log(b_t^{n-1}/b_t^n). \quad (4)$$

This implies a relationship between yields and forward rates:

$$r_t^n = \frac{1}{n} \sum_{k=1}^n f_t^k. \quad (5)$$

A simple version of the “expectations hypothesis” relates the forward rate to the expectation of a comparably timed future short rate, and a risk premium. In other words, the risk premium,  $rp_t^n$ , is defined by the equation

$$f_t^n \equiv E_t r_{t+n} + n\xi_t^n. \quad (6)$$

Combining equations (5) and (6) for the case of a 2-period bond,  $n=2$ , results in the familiar equation

$$r_t^2 = \frac{1}{2}(r_t + E_t r_{t+1}) + \xi_t. \quad (7)$$

Define the short-rate forecast error as  $\epsilon_{t+1}^r \equiv r_{t+1} - E_t r_{t+1}$ , and rewrite (7) as

$$r_{t+1} - r_t = 2(r_t^2 - r_t) - \xi_t - \epsilon_{t+1}^r. \quad (8)$$

When the risk premium is a constant, equation (8) forms a regression that can be estimated with observed bond-market behavior. There is also nothing particularly special about the 2-period maturity since we could imagine studying the comparable regression at any maturity for which we have data. The well-known Fama and Bliss (1987) empirical puzzle demonstrates that regressions based on equation (8) are strongly rejected in the data, and the coefficient on the term premium,  $r_t^2 - r_t$ , is significantly smaller than predicted by equation (8). These empirical facts have been established in a wide variety of subsequent studies summarized in Backus, Foresi, Mazumdar and Wu (2001). Dai and Singleton (2002) and (2003) study these rejections in the context of a wide variety of models of the risk premium,  $\xi_t$ .

The expectations hypothesis as stated, is not a very complete model since it neither specifies the stochastic process for exogenous shocks or the mapping from these shocks to endogenous bond prices and yields. By combining a specification of the risk-premium process with an interest-rate model, McCallum (1994) was able to integrate the expectations hypothesis and an analysis of a simple monetary policy rule that uses the short-rate as an instrument. In other words, he specified additional restrictions on the expectations hypothesis that embody an active monetary policy and an exogenous risk premium. We refer to the rule as the “McCallum Rule,” which takes the form

$$r_t = \mu_r r_{t-1} + 2\mu_f (r_t^2 - r_t) + \varepsilon_t, \quad (9)$$

where  $\varepsilon_t$  is a state variable summarizing the other exogenous determinants of monetary policy. The monetary policy rule implies that the monetary authority intervenes in the short-bond market to try to achieve two (perhaps conflicting) goals: “short-rate smoothing” governed by the parameter  $\mu_r$  and “yield-curve smoothing” governed by the parameter  $\mu_f$ . We will return to the motivation for and the practical implications of this monetary policy

rule shortly, but it is first instructive to see how the McCallum Rule affects our interpretation of the strong empirical rejections of the expectations hypothesis.

Combining equations (8) with (9) yields a linear stochastic difference equation for the interest rate:

$$E_t r_{t+1} = \left(\frac{1 + \mu_f}{\mu_f}\right)r_t - \frac{\mu_r}{\mu_f}r_{t-1} - \xi_t - \frac{1}{\mu_f}\varepsilon_t. \quad (10)$$

Using a first-order process for the risk premium,

$$\xi_t = \rho\xi_{t-1} + \epsilon_t^\xi, \quad (11)$$

where  $\epsilon_t^\xi$  is exogenous noise, and  $|\rho| < 1$ , McCallum (1994) shows that a stable solution, when it exists, is given by a linear function of the pre-determined or exogenous state variables,

$$r_t = M_0 + M_1 r_{t-1} + M_2 \xi_t + M_3 \varepsilon_t, \quad (12)$$

where

$$M_1 = \frac{1 + \mu_f - [(1 + \mu_f)^2 - 4\mu_f\mu_r]^{1/2}}{2\mu_f}, \quad (13)$$

is the equilibrium interest-rate-feedback coefficient which, in turn, determines the other coefficients:

$$M_0 = 0 \quad (14)$$

$$M_2 = \frac{2\mu_f}{1 + (1 - \rho - M_1)\mu_f} \quad (15)$$

$$M_3 = \frac{1}{1 + (1 - M_1)\mu_f}. \quad (16)$$

A particularly simple special case is extreme interest-rate smoothing,  $\mu_r = 1$  (which is the also the model studied by Kugler (1997)), which implies,

$$r_t = r_{t-1} + \frac{2\mu_f}{1 - \rho\mu_f}\xi_t + \varepsilon_t, \quad (17)$$

and an expectations-hypothesis-like regression based on the equation

$$E_t(r_{t+1} - r_t) = 2\rho\mu_f(r_t^2 - r_t), \quad (18)$$

which combines equation (17) with the risk-premium equation (11). It is evident, therefore, that the McCallum Rule combined with the expectations hypothesis can account for the Fama-Bliss type of empirical findings. The coefficient from a regression motivated by equation (8) must now be interpreted using the result in equation (18). The apparent downward bias is simply a reflection of the combination of persistence in the risk premium,  $\rho$ , and the monetary authority's yield-curve smoothing policy,  $\mu_f$ . Since it is reasonable to think of either or both of these parameters as numbers significantly less than 1, the downward bias documented in the empirical literature is a natural finding for this model.

Note also that if  $\rho = 0$  and there is no persistence in the risk premium, or if the monetary authority is unconcerned with the slope of the yield curve,  $\mu_f = 0$ , the model implies that there is nothing to be learned from the traditional expectations-hypothesis regression, as there is no longer a link between changes in the interest rate and the forward premium: the interest rate is engineered by the monetary authority to always follow a random walk, and the risk premium is simply unforecastable noise. Therefore, for McCallum's integration of monetary policy and a term-structure theory to be useful in rationalizing such empirical findings, persistence in the risk premium and sensitivity of monetary policy to the slope of the yield curve are central assumptions.

A limitation in McCallum's analysis is the exogeneity of the risk premium. The deeper source of the risk premium and how factors driving the risk premium might be related to factors that affect the interest rate, are left unspecified. Since McCallum's analysis, however, there have been numerous advances in the area of equilibrium term structure modelling that capture many of these effects. Moreover, as summarized by Dai and Singleton (2000, 2002, 2003) much of this literature has focused on linear rational expectations models (termed "affine models") and have been directed at similar empirical puzzles as those that motivated McCallum's work. To re-interpret McCallum's findings in the context of this newer class of models, we turn now to a log-linear model of multi-period bond pricing that anticipates the log-linear macroeconomic model in which we will imbed our analysis of monetary policy

rules. We adopt the Backus, Foresi, Mazumdar and Wu (2001) discrete-time version of the model of Duffie and Kan (1996) model of the term structure. The model begins with a characterization of the dynamic evolution of the state variables. Then the state variables are linked to the process for state prices, or the pricing kernel, which is then used to solve for arbitrage-free discount bond prices of all maturities.

Denote the  $k$  state variables of the model as the vector  $s_t$ . The dynamics of the state variables are modelled using a first-order vector autoregression with conditional volatility of the “square-root” form:

$$s_{t+1} = (I - \Phi)\theta + \Phi s_t + \Sigma(s_t)^{1/2}\epsilon_{t+1}, \quad (19)$$

where  $\epsilon_t \sim iid N(0, I)$ ,  $\Phi$  is a  $k \times k$  matrix of autoregressive parameters assumed to be stable,  $\theta$  is a  $k \times 1$  vector of drift parameters, and the conditional volatility process is given by:

$$\Sigma(s_t) = diag\{\alpha_i + \beta_i^\top s_t\}, i = 1, 2, \dots, k. \quad (20)$$

Since the variance must be positive, the parameters,  $\alpha_i$  and  $\beta_i$ , of the volatility process satisfy a set of sufficient conditions to insure this (see Backus, Foresi, Mazumdar and Wu (2001)).

The asset-pricing kernel is related to these state variables by the equation:

$$-\log m_{t+1} = \Gamma_0 + \Gamma_1^\top s_t + \lambda^\top \Sigma(s_t)^{1/2}\epsilon_{t+1}. \quad (21)$$

The  $k \times 1$  vector  $\lambda$  is referred to as the “price of risk.” The log-linear structure implies that the log of the pricing kernel inherits the conditional log normality of the state variable process.

We can use this pricing kernel to solve for arbitrage-free discount bond prices. By the definition of the pricing kernel,

$$b_t^n = E_t[m_{t+1}b_{t+1}^{n-1}]. \quad (22)$$

Given the log-normality built into the model, it is natural to conjecture a bond-price process that is log-linear in the state variables,  $s_t$ :

$$-\log b_t^n = A(n) + B(n)^\top s_t, \quad (23)$$

where  $A(n)$  and  $B(n)$  are  $k \times 1$  vectors of undetermined coefficients. Similarly, the continuously compounded yields will be linear functions of the state variables:

$$r_t^n = -n^{-1} \log b_t^n = \frac{A(n)}{n} + \frac{B(n)^\top}{n} s_t. \quad (24)$$

The bond-price/yield coefficients can be found recursively given the initial conditions  $A(0) = 0$  and  $B(0) = 0$ , (*i.e.*, the price of an instantaneous payment of 1 is 1). The recursions are given by:

$$A(n+1) = A(n) + \Gamma_0 + B(n)^\top (I - \Phi)\theta - \frac{1}{2} \sum_{j=1}^k (\lambda_j + B(n)_j)^2 \alpha_j, \quad (25)$$

and

$$B(n+1) = \Gamma_1^\top + B(n)^\top \Phi - \frac{1}{2} \sum_{j=1}^k (\lambda_j + B(n)_j)^2 \beta_j^\top. \quad (26)$$

The interest-rate process is given by:

$$\begin{aligned} r_t = -\log b_t^1 &= A(1) + B(1)s_t \\ &= \left( \Gamma_0 - \frac{1}{2} \sum_{j=1}^k \lambda_j^2 \alpha_j \right) + \left( \Gamma_1^\top - \frac{1}{2} \sum_{j=1}^k \lambda_j^2 \beta_j^\top \right) s_t \end{aligned} \quad (27)$$

Extending this to a 2-period-maturity bond will allow us to define the state-dependent risk premium,  $\xi_t$ , in a natural way. Define the risk premium using the expectations hypothesis as in equation (7):

$$\begin{aligned} r_t^2 &\equiv \frac{1}{2}(r_t + E_t r_{t+1}) + \xi_t \\ &= \frac{1}{2}[A(1) + B(1)^\top s_t + A(1) + B(1)^\top E_t s_{t+1}] + \xi_t \\ &= A(1) + \frac{1}{2}[B(1)^\top (I - \Phi)\theta + B(1)^\top (I + \Phi)s_t] + \xi_t. \end{aligned} \quad (28)$$

But since we also know that the 2-period yield satisfies the equilibrium pricing condition:

$$r_t^2 = \frac{A(2)}{2} + \frac{B(2)^\top}{2} s_t, \quad (29)$$

we can write, therefore, the risk premium on a 2-period bond as

$$\xi_t = -\frac{1}{2} \sum_{i=1}^k \hat{\Gamma}_i \left( \alpha_i + \beta_i^\top s_t \right), \quad (30)$$

where

$$\hat{\Gamma}_i \equiv -\frac{1}{2} \left( \Gamma_{1i} - \frac{1}{2} \left[ \sum_{j=1}^k \lambda_j^2 \beta_j^\top \right]_i \right) \left( \lambda_i + \frac{1}{2} \left( \Gamma_{1i} - \frac{1}{2} \left[ \sum_{j=1}^k \lambda_j^2 \beta_j^\top \right]_i \right) \right). \quad (31)$$

Naturally, this logic extends to the definition of a risk premium for any maturity bond.

The source of the state-dependent risk premium in this model is through the state-dependent conditional variance of the state variables and the pricing kernel. Absent this volatility, *i.e.*,  $\beta = 0$ , the risk premium is a constant function of the maturity and, hence, would provide no scope for an active policy response as characterized by the McCallum Rule. Dai and Singleton (2002) and (2003) have shown that a model in which the state dependence of the risk premium is the result of a state-dependent price of risk, rather than volatility, is equally tractable yet provides a much better empirical model. In light of these facts, we first develop the McCallum Rule and its relationship to the Taylor Rule in the stochastic volatility framework. Having developed the structure and the intuition that delivers the equivalence, we will be able to see how a state-dependent price-of-risk model can generate comparable results.

Relating the endogenous risk premium back to our earlier discussion of McCallum's model, the equilibrium risk premium in this model inherits the dynamics of the state variables,  $s_t$ . McCallum's specification for the dynamics of the risk premium therefore translates directly to our specification of the dynamics of the underlying state variables, provided the risk premium is state dependent,  $\beta_i \neq 0$ , for at least one value of  $i$ . In the next section we will relate these state variables to the shocks in a more complete macroeconomic model. Note, however, that we can repeat the McCallum Rule analysis for this more general term structure model which will allow us to characterize a similar sort of expectations-hypothesis-regression as in McCallum's analysis.

To see this most clearly, consider the case of a single-factor model,  $k = 1$ . Equations (30) and (19) imply that the dynamics of the risk premium in the one-factor model are:

$$\xi_{t+1} = -\frac{\hat{\Gamma}(\alpha + \beta\theta)(1 - \Phi)}{2} + \Phi\xi_t + -\frac{\hat{\Gamma}\beta}{2}\Sigma(s_t)^{1/2}\epsilon_{t+1}, \quad (32)$$

where, in obvious notation, all parameters are the natural scalar equivalents of the parameters in equation (19). Combining this risk-premium process with equation (10) allows us to solve for the equilibrium interest rate process as before:

$$r_t = \hat{M}_0 + \hat{M}_1 r_{t-1} + \hat{M}_2 \xi_t + \hat{M}_3 \varepsilon_t, \quad (33)$$

where

$$\hat{M}_1 = \frac{1 + \mu_f - [(1 + \mu_f)^2 - 4\mu_f\mu_r]^{1/2}}{2\mu_f}, \quad (34)$$

is the equilibrium interest-rate-feedback coefficient which, in turn, determines the other coefficients:

$$\hat{M}_0 = -\frac{\mu_f \hat{M}_2 \hat{\Gamma}(\alpha + \beta\theta)(1 - \Phi)}{2(1 - \mu_f \hat{M}_1)} \quad (35)$$

$$\hat{M}_2 = \frac{2\mu_f}{1 + (1 - \Phi - \hat{M}_1)\mu_f} \quad (36)$$

$$\hat{M}_3 = \frac{1}{1 + (1 - \hat{M}_1)\mu_f}. \quad (37)$$

In other words, with the exception of the constant,  $\hat{M}_0$ , and the replacement of  $\rho$  with  $\Phi$ , the solution is unchanged. Once again, turning to the special case of  $\mu_r = 1$ , we have

$$r_t = -\frac{\mu_f^2 \hat{\Gamma}(\alpha + \beta\theta)(1 - \Phi)}{(1 - \mu_f)(1 - \Phi\mu_f)} + r_{t-1} + \frac{2\mu_f}{1 - \Phi\mu_f} \xi_t + \varepsilon_t, \quad (38)$$

which, aside from a nonzero intercept term, implies an expectations-hypothesis regression comparable to equation (18):

$$E_t(r_{t+1} - r_t) = -\frac{\mu_f \hat{\Gamma}(\alpha + \beta\theta)(1 - \Phi)}{1 - \mu_f} + 2\Phi\mu_f(r_t^2 - r_t). \quad (39)$$

We next develop a macroeconomic model that both defines the abstract state variables,  $s_t$ , and provides the desired link between the two monetary policy rules under consideration.

### 3 A Macroeconomic Model

We introduce a simple New-Keynesian macroeconomic model. The model is an extension of the baseline macroeconomic model in Clarida, Galí and Gertler (1999). We extend their basic model to allow for time-varying risk, which leads to a time-varying risk premium in the term structure. The model consists of two equations: bond demand and an inflation relationship. Bond demand is modeled through the lifetime savings and investment problem of a representative agent and inflation is modeled through firms' staggered price setting with cost push shocks. Monetary policy is modeled by an interest rate rule.

There is a representative consumer who chooses consumption and investment plans to solve the intertemporal optimization problem:

$$\max E_t \left[ \sum_{i=0}^{\infty} \exp\{-\delta i\} \frac{C_{t+i}^{1-\gamma}}{1-\gamma} \right], \quad (40)$$

subject to the standard intertemporal budget constraint. Here  $C_t$  is consumption,  $\exp\{-\delta\}$  is the time preference parameter and  $\gamma$  is the coefficient of relative risk aversion.

Let  $P_t$  be the nominal price level at time  $t$  and let  $r_t$  be the continuously compounded interest rate. The first-order condition for one-period bond holding is:

$$\exp\{-r_t\} = \exp -\delta E_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+1}} \right]. \quad (41)$$

Similar first-order conditions apply for the holdings of all financial securities. The logarithmic pricing kernel therefore is

$$-\log m_{t+1} = \delta + \gamma (\Delta c_{t+1}) + \pi_{t+1}, \quad (42)$$

where  $c_t \equiv \log C_t$  is the logarithm of consumption,  $\Delta$  is the difference operator and  $\pi_{t+1} \equiv \Delta \log P_{t+1}$  is the inflation rate.

Let  $z_t$  be the logarithm of the natural rate of output, and  $y_t$  the logarithm of the actual rate of output. The logarithm of the output gap,  $x_t$ , is

$$x_t = y_t - z_t. \quad (43)$$

Let  $G_t$  be government expenditures. The aggregate budget constraint for the economy is

$$C_t + G_t = Y_t. \quad (44)$$

Defining  $g_t \equiv -\log\left(1 - \frac{G_t}{Y_t}\right)$  as relative government spending, and  $\hat{z}_t$  as the natural rate of output less relative government spending,  $\hat{z}_t \equiv z_t - g_t$ , the logarithmic form of the aggregate resource constraint is

$$c_t = x_t + \hat{z}_t. \quad (45)$$

The natural rate of output less relative government spending,  $\hat{z}_t$ , is determined exogenously. The output gap,  $x_t$ , is determined endogenously by the interest rate policy set by the monetary authority.

Inflation evolves according to

$$\pi_t = \psi x_t + \kappa E_t \pi_{t+1} + u_t. \quad (46)$$

where  $\psi$  is a positive constant measuring the impact of the current output gap on inflation and  $0 < \kappa < 1$  is the effect of expected future inflation on current inflation. Equation (46) is derived from a model of firms' optimal price setting decision with staggered price setting. Here,  $u_t$  is a stochastic shock to firms' marginal costs, and we refer to  $u_t$  as the cost push shock. Iterating equation (46) forward, the equilibrium inflation rate is

$$\pi_t = \sum_{i=0}^{\infty} \kappa^i (\psi E_t [x_{t+i}] + E_t [u_{t+i}]). \quad (47)$$

Using the aggregate resource constraint, equation (45), the definition of  $\hat{z}_t$ , and the inflation process equation (47), the logarithmic pricing kernel is:

$$-\log m_{t+1} = \delta + \gamma (\Delta x_{t+1} + \Delta \hat{z}_{t+1}) + \sum_{i=0}^{\infty} \kappa^i (\psi E_{t+1} [x_{t+1+i}] + E_{t+1} [u_{t+1+i}]). \quad (48)$$

Both  $\Delta \hat{z}_t$  and  $u_t$  evolve exogenously. To describe the conditional volatility of the cost push shock,  $u_t$ , we introduce an additional state variable  $\eta_t$ . The state variable  $\eta_t$  can

help predict the conditional volatility of cost push shocks, and therefore contributes to the dynamics of the risk premium in the term structure.

To parallel the structure of the term structure model of Section 2, define the state vector  $s_t^\top \equiv (\Delta \hat{z}_t, \eta_t, u_t)$ . The vector  $s_t$  follows an autoregressive process with volatility of the “square root” form:

$$s_{t+1} = (1 - \Phi)\theta + \Phi s_t + \Sigma(s_t)^{1/2} \epsilon_{t+1}, \quad (49)$$

with

$$\Phi \equiv \text{diag}\{\Phi_z, \Phi_\eta, \Phi_u\}, \quad (50)$$

$$\theta \equiv [\theta_z, \theta_\eta, \theta_u], \quad (51)$$

$$\Sigma(s_t) = \text{diag}\{\alpha_z, \alpha_\eta, \alpha_u + \beta_{u\eta}\eta_t + \beta_{uu}u_t\} \quad (52)$$

and  $\epsilon \sim iid N(0, I)$ .

The conditional volatilities of the growth of the natural rate of output and  $\eta_t$  are both constant while the conditional volatility of the cost push shock can be time varying. The intuition of our results, however, is robust to incorporating time varying volatility in the growth of the natural rate of output or the shock  $\eta_t$ . Our current parameterization is chosen for simplicity.

The dynamics of the risk premium depend only on the dynamics of the conditional volatility of the cost push shock. If  $\beta_{u\eta} = \beta_{uu} = 0$  then the conditional volatility of all the state variables is constant and there is no state dependence in the risk premium in the term structure in equilibrium. If  $\beta_{u\eta} = 0$  and  $\beta_{uu} \neq 0$  the conditional volatility of the cost push shock only depends on the current level of the cost push; here the risk premium only depends on the cost push shock. If  $\beta_{u\eta} \neq 0$  and  $\beta_{uu} = 0$  the conditional volatility of cost push shocks only depends on the current level of  $\eta$ ; the risk premium only depends on  $\eta$ .

Both the output gap and the inflation rate are determined endogenously. The monetary authority sets the interest rate to respond to the current value of the state variables. The

current output gap adjusts so that the the equilibrium bond demand derived from equation (41) holds. Inflation is set according to equation (47). Rational expectations holds in equilibrium; the representative agent's beliefs about the distribution of future output gaps and future inflation rates are consistent with the monetary authority's policy rule and the process followed by the state variables.

We assume that the monetary authority sets an interest rate policy that is an affine function of the state vector. We will show in Proposition 1 that a version of such an interest rate rule implies that the current output gap and current inflation are linearly related to the cost push shock  $u_t$ . If we use the framework developed by Clarida, Galí and Gertler (1999) to determine optimal monetary policy, the optimal output gap would depend only on the cost push shock — see their equation (3.5), for example.

The monetary authority sets an interest rate policy such that the output gap is

$$x_t = D_u u_t, \tag{53}$$

where  $D_u$  is a constant determined by the monetary policy.

Using the dynamics of the state variables and the output gap process, the inflation process is

$$\pi_t = \frac{\psi D_u + 1}{1 - \kappa \Phi_u} (1 - \Phi_u) \theta_u + \frac{\psi D_u + 1}{1 - \kappa \Phi_u} u_t. \tag{54}$$

Using the output gap and the solution for the inflation rate, we can compute the logarithm of the pricing kernel, which leads us to our first set of results, summarized in the following proposition.

**Proposition 1** *Suppose that the output gap is a linear function of the cost push shock, with coefficient  $D_u$ . Then, the pricing kernel is*

$$-\log m_{t+1} = \Gamma_0 + \Gamma_1^\top s_t + \lambda^\top \Sigma(s_t)^{1/2} \epsilon_{t+1}, \tag{55}$$

with

$$\Gamma_0 = \delta + \left[ \gamma, 0, \gamma D_u + \frac{\psi D_u + 1}{(1 - \kappa \Phi_u)(1 - \kappa)} \right] (I - \Phi)\theta, \quad (56)$$

$$\Gamma_1^\top = \left[ \gamma \Phi_z, 0, \left( \gamma D_u + \frac{\psi D_u + 1}{(1 - \kappa \Phi_u)} \right) \Phi_u - \gamma D_u \right], \quad (57)$$

and

$$\lambda^\top = \left[ \gamma, 0, \gamma D_u + \frac{\psi D_u}{1 - \kappa \Phi_u} \right]. \quad (58)$$

The interest rate is equal to

$$r_t = F_0 + F_1^\top s_t, \quad (59)$$

with

$$F_1^\top = \left[ \gamma \Phi_z, -\frac{D_u}{2} \left( \gamma + \frac{\psi}{1 - \kappa \Phi_u} \right) \beta_{u\eta}, \left( \gamma D_u + \frac{\psi D_u + 1}{1 - \kappa \Phi_u} \right) \Phi_u - \gamma D_u - \frac{D_u}{2} \left( \gamma + \frac{\psi}{1 - \kappa \Phi_u} \right) \beta_{uu} \right]. \quad (60)$$

The risk premium is equal to

$$rp_t = H_0 + H_1^\top s_t, \quad (61)$$

with

$$H_1^\top = -\frac{1}{4} F_{1u} (F_{1u} + 2\lambda_u) [0, \beta_{u\eta}, \beta_{uu}], \quad (62)$$

where  $F_{1u}$  is last element of  $F_1$  and  $\lambda_u$  is the last element of  $\lambda$ .

*Conversely, if the monetary authority sets interest rates according to equation (59), then the output gap and inflation rate follow as in equations (53) and (54).*

**Proof:** The solutions for the pricing kernel follows by substituting the output gap and inflation solutions into the pricing kernel, equation (48). The resulting interest rate and risk premium follow from the expressions for the affine model developed in Section 2. The converse follows from inverting the bond demand equation for the current output gap, given a linear form for expected output and expected inflation and matching terms. ■

The proposition shows that a linear feedback rule for the interest rate can result in an output gap that is linear in the cost push shock. We can also rearrange the interest rate rule in the form of a “forward-looking Taylor rule” of the type studied by Clarida, Galí and Gertler (1999)

$$r_t = F_0 - F_{1u} \frac{(1 - \kappa\Phi_u)(1 - \Phi_u)\theta_u}{\kappa\Phi_u(1 - \alpha\Phi_u)} + F_{1z}\Delta\hat{z}_t + F_{1\eta}\eta_t + F_{1u} \frac{1 - \kappa\Phi_u}{\Phi_u(\psi D_u + 1)} E_t[\pi_{t+1}] \quad (63)$$

The proposition shows that a linear feedback rule for the interest rate results in an affine term structure model. The interest rate can be put in the form of a McCallum Rule. We consider two cases. In the first case, suppose that the conditional volatility of the cost push shock depends on  $\eta_t$  only:  $\beta_{u\eta} \neq 0$  and  $\beta_{uu} = 0$ . To write the interest rate in the form of the McCallum Rule, define the matrix:

$$\Lambda = \begin{bmatrix} F_1^\top \\ H_1^\top \\ (0, 0, 1) \end{bmatrix}, \quad (64)$$

and the vector

$$\Xi_t^\top = [r_t, \xi_t, \Delta\hat{z}_t]. \quad (65)$$

Then,

$$\begin{aligned} \Xi_{t+1} &= \Lambda s_t \\ &= \Lambda(1 - \Phi)\theta + \Lambda\Phi\Lambda^{-1}\Xi_t + \Lambda\Sigma(s_t)^{1/2}\epsilon_{t+1}. \end{aligned} \quad (66)$$

The parameters in the McCallum Rule are found by matching parameters for  $r_{t+1}$  in the equation (66) – the first row of the matrix  $\Lambda\Phi\Lambda^{-1}$  – with the parameters in equation (38). Here,  $[\Lambda\Phi\Lambda^{-1}]_{13} \times \Delta\hat{z}_t$  is the exogenous parameter affecting the monetary policy in the McCallum rule; the term  $\epsilon_t$  in equation (38).

We have therefore shown that when the conditional volatility of the cost push shocks depend on  $\eta_t$ , the interest rate rule depends on lagged interest rates, the risk premium and

$\Delta\hat{z}_t$ . As a consequence, the McCallum Rule can implement a policy where the output gap depends on the cost push shock. We have not, however, shown that empirically plausible parameters for the structural model and the interest rate policy rule would lead to parameterization for the McCallum Rule that is consistent with empirical evidence on deviations from the expectations hypothesis.

For the second case, suppose that the conditional volatility of the cost push shock depends on  $u_t$  only:  $\beta_{u\eta} = 0$  and  $\beta_{\eta u} = 0$ . Here, the risk premium only depends on  $u_t$ , implying that the interest rate only depends on  $\Delta z_t$  and  $u_t$ . Although it is possible for the interest rate to be written in the form of a McCallum rule, now the coefficient on  $u_t$  satisfy two restrictions. First, the output gap should react to the cost push shock appropriately. Second, the interest rate must react to the risk premium appropriately. There is nothing in the model that guarantees that such conditions can be met. We will explore the quantitative implications of the model in the next section.

## 4 Quantitative Implications

Here, we study plausibly parameterized versions of the macro model and the policy rule. Use the examples to determine what properties the implied output gap rule and term structure have in an economy satisfying a McCallum Rule.

*To be added.*

## 5 Conclusion

We have shown that the McCallum (1994a) result that the expectations hypothesis, when adjusted for an active interest-rate monetary policy rule that has a yield-curve smoothing component, matches observed dynamic patterns in the term structure than the expectations hypothesis alone, extends to the case of an endogenous risk premium. In addition, we have

shown that a simple New-Keynesian macroeconomics model along the lines of Clarida, Galií, and Gertler (1999), can be used as a macroeconomic foundation for identifying the relevant state variables and parameters of a latent-variable or unobservable-factor model of the term structure. Within this macro-term structure model, we show when the McCallum Rule is equivalent to the Taylor Rule. The equivalence depends critically on the macroeconomic source of the volatility-driven risk premium in long-term bonds. Further, we establish conditions on the macroeconomic model under which the McCallum Rule could be viewed as an optimal monetary policy. If there is an exogenous factor that affects the risk premium on the long bond but is unrelated to the inflation-output gap tradeoff, then the McCallum Rule can implement the optimal policy. That is, it accommodates the risk premium that acts as a wedge in the relationship between the current output gap and interest rates. If the risk premium is related to the inflation-output gap tradeoff, then it is difficult for the McCallum Rule to simultaneously target the risk-premium wedge and the output gap. This may be the reason why Cochrane and Piazzesi (2002) find that allowing both the long rate and the long-short spread to enter the policy rule for target-rate changes, provides the best empirical model. In the context of the McCallum Rule, such a policy function allows the monetary authority to balance the inflation-output gap tradeoff with the risk-premium wedge.

Future research will extend our stochastic-volatility-based risk-premium model to the class of models shown by Dai and Singleton (2002) to hold more promise for capturing all of the salient features of observed yield-curve behavior. Such a model will capture the state dependence in the risk premium through a state-dependent price of risk ( $\lambda$  in equation (21)), rather than through state-dependent volatility. Such a model is as tractable as the model we study, but has better empirical properties. The macroeconomic foundations of this model will maintain constant volatilities, but will require a preference shock in equation (41) which can be motivated in a variety of ways (*e.g.*, the external habits specification of Campbell and Cochrane (1999), Wachter (2004) or Dai (2000)). Mirroring the result in section (3) of our paper, the factors driving this preference shock must be correlated with

the inflation-output gap tradeoff to provide scope for the McCallum Rule or the Taylor Rule to capture optimal monetary policy.

## References

- Ang, A. and M. Piazzesi. (2003). “A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables,” **Journal of Monetary Economics**, 50, May, 745–787.
- Ang, A., S. Dong and M. Piazzesi. (2004). “No-Arbitrage Taylor Rules,” Working Paper, Columbia University.
- Backus, D., S. Foresi, A. Mozumdar and L. Wu. (2001). “Predictable Changes in Yields and Forward Rates,” **Journal of Financial Economics**, 59, 281–311.
- Campbell, J.Y. and J. H. Cochrane. (1999). “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior,” **Journal of Political Economy**, 107, 205–51.
- Clarida, R., J. Galí and M. Gertler. (1999). “The Science of Monetary Policy: A New Keynesian Perspective,” **Journal of Economic Literature**, 37, 1661–1707.
- Clarida, R., J. Galí and M. Gertler. (2000). “Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory,” **The Quarterly Journal of Economics**, 115, 147–180.
- Cochrane, J. and M. Piazzesi. (2002). “The Fed and Interest Rates – A High-Frequency Identification,” **American Economic Review, Papers and Proceedings**, 92, May, 90–95.
- Dai, Q. (2000). “From Equity Premium Puzzle to Expectations Puzzle: A General Equilibrium Production Economy with Stochastic Habit Formation,” Working Paper, Stern School of Business, New York University.
- Dai, Q. and K. Singleton. (2000). “Specification Analysis of Affine Term Structure Models,” **Journal of Finance**, 55, 1943–1978.
- Dai, Q. and K. Singleton. (2002). “Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure,” **Journal of Financial Economics**, 63, 415–441.
- Dai, Q. and K. Singleton. (2003). “Term Structure Dynamics in Theory and Reality,” **Review of Financial Studies**, 16, 631–678.
- Duffie, D. and R. Kan. (1996). “A Yield-Factor Model of Interest Rates,” **Mathematical Finance**, 6, pp. 379–406.
- Estrella, A. and F. S. Mishkin. (1997). “The Predictive Power of the Term Structure of Interest Rates in Europe and the United States: Implications for the European Central Bank,” **European Economic Review**, 41, 1375–1401.
- Evans, C. L. and D. A. Marshall. (1998). “Monetary Policy and the Term Structure of Nominal Interest Rates: Evidence and Theory,” **Carnegie-Rochester Conference Series on Public Policy**, 49, 53–111.
- Evans, C. L. and D. A. Marshall. (2001). “Economic Determinants of the Term Structure of Nominal Interest Rates,” Working paper, Federal Reserve Bank of Chicago, Chicago, IL.

- Fama, E. F. and R. R. Bliss. (1987). “The Information in Long-Maturity Forward Rates,” **The American Economic Review**, 77, September, 680–692.
- Goodfriend, M. (1991). “Interest Rates and the Conduct of Monetary Policy,” **Carnegie-Rochester Conference Series on Public Policy**, 34, 7–30.
- Goodfriend, M. (1993). “Interest Rate Policy and the Inflation Scare Problem: 1979–1992,” **Federal Reserve Bank of Richmond Economic Quarterly**, 79, Winter, 1–23.
- Kugler, P. (1997). “Central Bank Policy Reaction and the Expectations Hypothesis of the Term Structure,” **International Journal of Financial Economics**, 2, 217–224.
- Mankiw, N. G. and J. A. Miron. (1986). “The Changing Behavior of the Term Structure of Interest Rates,” **The Quarterly Journal of Economics**, 101, 211–228.
- McCallum, B. T. (1994a). “Monetary Policy and the Term Structure of Interest Rates,” NBER Working Paper No. 4938.
- McCallum, B. T. (1994b). “A Reconsideration of the Uncovered Interest Rate Parity Relationship,” **Journal of Monetary Economics**, 33, 105–132.
- Piazzesi, M. (2001). “An Econometric Model of the Yield Curve with Macroeconomic Jump Effects,” NBER Working Paper No. 8246.
- Piazzesi, M. and E. Swanson. (2004). “Futures Prices as Risk-Adjusted Forecasts of Monetary Policy,” NBER Working Paper No. 10547.
- Rudebusch, G. (1998). “Do Measures of Monetary Policy in a VAR Make Sense?” **International Economic Review**, 39, 907–31.
- Rudebusch, G. and T. Wu. (2004). “A Macro-Finance Model of the Term Structure, Monetary Policy, and the Economy,” Working Paper, Federal Reserve Bank of San Francisco.
- Taylor, J. B. (1993). “Discretion versus Policy Rules in Practice,” **Carnegie-Rochester Conference Series on Public Policy**, 39, December, 195–214.
- Wachter, J. (2004). “A Consumption-Based Model of the Term Structure of Interest Rates,” Working Paper, Wharton School of Business, University of Pennsylvania.