Abstract
We inject uncertainty — risk and ambiguity — into an otherwise standard business cycle model and describe the consequences for business cycles. We find that increases in uncertainty in this model do not account, in this model, for either the magnitude or the persistence of the most recent recession. We speculate about extensions that might do better along one or both dimensions.

JEL Classification Codes: E32, D81, G12.

Keywords: uncertainty; smooth ambiguity; certainty equivalent; recursive preferences; pricing kernel; asset returns; learning.

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1 Introduction

The most recent recession in the United States was the most severe since the 1930s, and it’s been followed by an unusually long recovery. We see how this played out in Figures 1 to 3, taken from Bethune, Cooley, and Rupert (2014). The figures illustrate the recession and recovery, relative to the previous peak, in output, consumption, and investment over the last five recessions. Output, for example, hit a cyclical peak in the fourth quarter of 2007, declined by about 4 percent over the next 6 quarters, and only surpassed its previous peak 14 quarters later (Figure 1). The depth and especially the length are greater than in any of the earlier recessions. Consumption (Figure 2) and investment (Figure 3) show broadly similar behavior. Consumption fell 3 percent and took 12 quarters to return to its previous level. Nonresidential fixed investment fell almost 20 percent and took 24 quarters (6 years!) to recover.

These patterns are familiar to all of us. The relative magnitudes are what we tell our students: consumption falls less than output, and investment much more. The same with the comovements: all three variables, and many more besides, declined together. The only somewhat unusual feature here is the slow recovery of investment.

Since the patterns are familiar, we might expect to account for them with some variant of the Kydland-Prescott (1982) model, in which declines in productivity generate precisely this collection of facts. The unusually large magnitude of the recession would reflect, in this model, an unusually large drop in productivity. The problem with this account is that measured productivity barely fell. We could document this in a number of ways, but the simplest is to measure productivity by the ratio of output to hours worked. We see the result in Figure 4.

But if productivity isn’t the source of the last recession, what is? Yes, we know, there was a financial crisis. But what shocks — or wedges — do we need to add to the model to reproduce its effects? We have no shortage of candidates, but the leading one is “uncertainty.” Among the many suggestions to this effect is a wonderful comment by Marco Buti, Director General of the European Commission (Buti, 2012): “Economic theory suggests that uncertainty has a detrimental effect on economic activity by giving agents the incentive to postpone investment, consumption and employment decisions until uncertainty is resolved, and by pushing up the cost of capital through increased risk premia.” Nick Bloom (2013), who has written extensively on uncertainty, adds: “The onset of the Great Recession was accompanied by a massive surge in uncertainty. The size of this uncertainty shock was so large it potentially accounted for around one third of the 9% drop in GDP versus trend during 2008-2009.”

With this motivation, we study uncertainty in a streamlined business cycle model and ask: Can a sharp increase in uncertainty account for the magnitude or persistence of the last recession? How does uncertainty affect the dynamics of consumption, investment, and output? Does it interfere with traditional business cycle comovements, in which all three of
these variables move up and down together? Can uncertainty account for differences in the responses of consumption and investment during the latest recovery? For other features of business cycles?

To answer these questions, we inject uncertainty into a simple business cycle model, a streamlined version of Kydland and Prescott (1982) with constant labor supply. We add three ingredients to their model: recursive preferences, a unit root in the productivity process, and several kinds of uncertainty. Recursive preferences are a natural generalization of the additive preferences used in most business cycle models. A large body of work suggests that their extra flexibility is helpful in accounting for asset prices (Bansal and Yaron, 2004, for example) but has little impact on the behavior of macroeconomic quantities (Tallarini, 2000). In this respect they differ markedly from habit-based preferences, which affect both quantities and asset prices. The unit root is essential to delivering realistic risk premiums. Without it, the asset with the largest risk premium is a long-maturity bond (Alvarez and Jermann, 2005, Proposition 2), which is contradicted by the evidence.

The most important new ingredient is the third one: uncertainty. We consider stochastic processes for the conditional mean and variance of productivity growth and the uncertainty that arises endogenously from learning. Preferences play a role here in the representative agent’s responses to these sources of uncertainty. Typically greater aversion to uncertainty leads to stronger responses of consumption to variation in uncertainty. We consider preferences toward risk, in which the distribution of outcomes is understood by our representative agent, and ambiguity, in which it is not.

We use these ingredients to make two contributions. The first is to assess the impact of uncertainty on macroeconomic fluctuations and address what we call the Barro-King problem. Barro and King (1984) showed that shocks to anything but productivity generate counterfactual comovements. Uncertainty, of course, is one such shock. In our model, a change in uncertainty drives consumption and investment in opposite directions. If the effect is strong enough, we lose the strong procyclical movements of these variables that we see in the data. We show, however, that in models with quasi-realistic parameter values, the productivity shocks usually dominate. In fact, the modest decline in the correlation of consumption and investment with output may bring the model closer to the evidence.

Our second contribution is to describe precisely how uncertainty affects decisions. We compute properties of models with accurate numerical procedures, but we gain insight from loglinear approximations analogous to Campbell’s (1994). We think the approximations give us clarity about how uncertainty works that would be hard to come by otherwise. And despite rumors to the contrary, such loglinear approximations are compatible with uncertainty.

With these tools we find:

- Tallarini property. Tallarini (2000) showed that in a model with an intertemporal elasticity of substitution (IES) of one, iid productivity growth, and constant risk, the behavior
of quantities in a model with recursive preferences is the same as in one with additive preferences and the same IES. Recursive preferences affect asset prices, but not consumption, investment, and output. We extend his result to a model with arbitrary IES and arbitrary linear dynamics in productivity growth. Our result applies to a loglinear approximation of the model, but we find that the approximation is close to the more accurate solution.

• Uncertainty. Consumption and investment decisions are functions of the state, which here includes a state variable representing uncertainty. We find that consumption typically falls when uncertainty rises, although there are parameter configurations in which the reverse is true. The magnitude of the effect depends on both the IES and risk aversion. In this respect, recursive preferences play a quantitatively important role in the transmission of uncertainty to the economy. These opposite movements are too small to have much effect on the business cycle properties of the model, including its comovements, so we pass the Barro-King test.

• Separation property. The impact of uncertainty on dynamics is limited by what we call the separation property: the internal dynamics of the capital stock are independent of uncertainty and its properties. More precisely, the response of consumption and next period’s capital stock to today’s capital stock is independent of the shock and its properties. This is a standard feature of linear-quadratic models. It also applies to our loglinear approximations and, to a close approximation, to numerical solutions of our models. As a result, uncertainty in this model cannot account for an unusually slow recovery except through the shock.

• Ambiguity. Ambiguity provides a rationale for strong uncertainty aversion to features of the model that are important to aggregate welfare yet hard to observe. One of these is the conditional mean of productivity growth. Information about the conditional mean affects consumption and, eventually, capital and output. The sign and magnitude of the impact depend on the IES and ambiguity aversion.

So what’s the bottom line? The recursive business cycle model provides a mechanism through which fluctuations in uncertainty affect the dynamics of aggregate quantities. These fluctuations can have a measurable impact on the magnitude of variation in growth rates of output, consumption, and investment, and on the correlations among them. With most parameter settings, this impact is small. We also find that uncertainty has essentially no impact on the internal dynamics of the model: an increase in uncertainty produces a more persistent decline in (say) consumption only if uncertainty is itself persistent.

A few words on notation: We use a number of conventions to keep the analysis as simple as we can make it, which is unfortunately not all that simple. (i) For the most part, Greek letters are parameters and Latin letters are variables or coefficients. (ii) We use τ subscripts (x_t, for example) to represent random variables and the same letters without subscripts (x) to represent their means. Or, more commonly, log x represents the mean of log x_t rather than the log of the mean of x_t. (iii) We also use τ subscripts to denote dependence of a function on the state. Thus f(x_t) might be denoted f_t. (iv) We use variable subscripts
to denote derivatives; for example, \( f_{xt} = \partial f(x_t)/\partial x_t \). (v) The abbreviation iid means independent and identically distributed and NID\((a, b)\) means normally and independently distributed with mean \(a\) and variance \(b\).

# 2 Risk

We approach uncertainty from the perspective of decision theory. We use the term risk to describe random environments in which the distribution of outcomes is known. We use ambiguity to describe environments in which some aspect of the distribution is unknown. Uncertainty is an umbrella term that includes both risk and ambiguity. We consider risk here, and turn to ambiguity in Section 6.

## 2.1 Risk preference in static environments

Our treatment of risk is standard in macroeconomics and finance: the distribution over outcomes is known (risk) and equal to the distribution that generates the data (rational expectations).

We model preference over risky outcomes with expected utility. Consider a static environment with a random state \(s\) and consumption \(c(s)\) defined over it. Risk is a known nonconstant probability distribution over \(s\), which induces a distribution over \(c\). We characterize behavior toward risk with a certainty equivalent function, which transforms utility back into consumption units. More formally, the certainty equivalent \(\mu(c)\) is the level of constant consumption that delivers the same utility. If \(c\) is constant, then \(\mu(c) = c\). If \(c\) is risky, then risk aversion is indicated by \(\mu(c) < E(c)\). A number of common certainty equivalent functions with this feature are described in Backus, Routledge, and Zin (2005, Section 3). We refer to the log difference \(\log E(c) - \log \mu(c) > 0\) as a risk adjustment.

We rely exclusively on the expected utility certainty equivalent,

\[
\mu(c) = u^{-1}[E(u(c))],
\]

for some increasing concave function \(u\). The standard example in macroeconomics and finance is power utility: \(u(c) = c^\alpha/\alpha\). Here \(\alpha < 1\) and \(1 - \alpha > 0\) is commonly referred to as the coefficient of relative risk aversion. The power utility certainty equivalent is

\[
\mu(c) = [E(c^\alpha)]^{1/\alpha},
\]

which is conveniently homogeneous of degree one.

**Example.** It’s not essential, but we build most of our examples on the lognormal distribution. Let \(\log c = s \sim \mathcal{N}(\kappa_1, \kappa_2)\) (the log of consumption is normal with mean \(\kappa_1\) and variance \(\kappa_2\)). The log of the moment generating function for \(s\) is \(\log E(e^{\theta s}) = \log E(e^{\theta}) = \theta \kappa_1 + \theta^2 \kappa_2/2\). Therefore \(\log E(c) = \kappa_1 + \kappa_2/2\), \(\log E(c^\alpha) = \alpha \kappa_1 + \alpha^2 \kappa_2/2\), and \(\log \mu(c) = \kappa_1 + \alpha \kappa_2/2\). Risk aversion is implied by the risk adjustment \(\log E(c) - \log \mu(c) = (1 - \alpha)\kappa_2/2 > 0\).
2.2 Risk preference in dynamic environments

We extend risk preference to dynamic environments with the recursive technology developed by Kreps and Porteus (1978). Utility \( U_t \) from date \( t \) on has the form

\[
U_t = V[c_t, \mu_t(U_{t+1})].
\]  

(1)

Time preference is built into the time aggregator \( V \) and risk preference is built into the certainty equivalent \( \mu_t \).

We assume throughout that the time aggregator \( V \) and certainty equivalent \( \mu_t \) are homogeneous of degree one, which allows us to use them in environments that are stationary in growth rates. We use the constant elasticity time aggregator suggested by Epstein and Zin (1989),

\[
V[c_t, \mu_t(U_{t+1})] = [(1 - \beta)c_t^\rho + \beta \mu_t(U_{t+1})^\rho]^{1/\rho},
\]

(2)

with \( 0 < \beta < 1 \) and \( \rho < 1 \). Here \( \sigma = 1/(1 - \rho) \) is the intertemporal elasticity of substitution: the elasticity of substitution between current consumption and the certainty equivalent of future utility. Like certainty equivalents, this recursive representation of preferences expresses utility in consumption units. Consider a constant consumption path \( c_{t+j} = c \) for all \( j \geq 0 \). Then \( U_t = U_{t+1} = \mu_t(U_{t+1}) = c \). Differences of \( U_t \) from current consumption \( c_t \) reflect some combination of timing and uncertainty in the path of future consumption. Preferences toward risk and ambiguity are built into the certainty equivalent \( \mu_t \).

In the Markov environments we study, outcomes at date \( t \) are functions of the state \( s_t \). We denote dependence on the state with a \( t \) subscript. Thus current utility is \( U(s_t) = U_t \) and next-period utility is \( U(s_{t+1}) = U_{t+1} \). Similarly, the conditional probability of a succeeding state \( s_{t+1} \) is \( \pi(s_{t+1}|s_t) = \pi_t(s_{t+1}) \) and the conditional expectation computed from this probability is \( E_t \). Risk is captured with the certainty equivalent

\[
\mu_t(U_{t+1}) = \left[E_t(U_{t+1})^\alpha\right]^{1/\alpha}
\]

(3)

with, as before, risk parameter \( \alpha < 1 \). The coefficient of relative risk aversion with respect to risk in future utility is \( 1 - \alpha \).

3 A recursive business cycle model

We describe a simple business cycle model, based loosely on Kydland and Prescott (1982), and the Bellman equation associated with its solution. We scale the model to take into account growth in productivity. We also show how some of the basic properties of risk premiums in the same model can be derived from the representative agent’s intertemporal marginal rate of substitution.
3.1 Model

Our benchmark model starts with recursive preferences: equation (1) with homogeneous of degree one time aggregator $V$ and certainty equivalent function $\mu$. Production uses capital $(k_t)$ and labor $(n_t)$ inputs and leads to the law of motion

$$k_{t+1} = f(k_t, a_t n_t) - c_t,$$

(4)

where $f$ is also homogeneous of degree one and $a_t$ is (labor) productivity. In examples, we fix the labor input at one $(n_t = 1)$ and use a constant elasticity production function with constant depreciation:

$$f(k_t, a_t n_t) = \left[\omega k_t^\nu + (1-\omega)(a_t n_t)^\nu\right]^{1/\nu} + (1-\delta)k_t = y_t + (1-\delta)k_t,$$

(5)

where $0 < \delta \leq 1$ is the depreciation rate, $\nu < 1$, $1/(1-\nu)$ is the elasticity of substitution between capital and labor, and $y_t$ is output. Investment is $i_t = y_t - c_t$.

The source of fluctuations in this model is a stochastic process for productivity growth. We use a loglinear process for convenience and introduce stochastic volatility so that the model has variation over time in uncertainty. Specifically, productivity growth $g_t = a_t / a_{t-1}$ is tied to a state vector $x_t$ by

$$\log g_t = \log g_t + e^\top x_t,$$

where $e$ is an arbitrary vector of coefficients. The vector $x_t$ has a state-space structure,

$$x_{t+1} = Ax_t + B v_t^{1/2} w_{1t+1},$$

(6)

with $\{w_{1t}\} \sim \text{NID}(0, I)$. If $A = [0]$ then the conditional mean of $\log g_t$ is constant. Otherwise $Ax_t$ adds a predictable component of future productivity, what you might call “news.” The conditional variance $v_t$ is AR(1):

$$v_{t+1} = (1-\varphi)v_t + \varphi v_t + \tau w_{2t+1},$$

(7)

with $\{w_{2t}\} \sim \text{NID}(0, 1)$ and independent of $\{w_{1t}\}$.

We find a competitive equilibrium as the solution to a planning problem: maximize utility subject to the laws of motion for the state $s_t = (k_t, a_t, x_t, v_t)$. The Bellman equation might be expressed then as

$$J(k_t, a_t, x_t, v_t) = \max_{c_t} V\{c_t, \mu_t[J(k_{t+1}, a_{t+1}, x_{t+1}, v_{t+1})]\},$$

subject to the laws of motion (4), $a_{t+1} = a_t g_{t+1} = a_t \exp(\log g + e^\top x_{t+1})$, (6), and (7).

Since $V$, $\mu$, and $f$ are all homogeneous of degree one, $J$ is homogeneous of degree one in $k_t$ and $a_t$. That allows us to divide the Bellman equation by $a_t$ and express the problem in terms of scaled variables, $\tilde{k}_t = k_t / a_t$ and $\tilde{c}_t = c_t / a_t$. The scaled Bellman equation is

$$J(\tilde{k}_t, 1, x_t, v_t) = \max_{\tilde{c}_t} V\{\tilde{c}_t, \mu_t[g_{t+1} J(\tilde{k}_{t+1}, 1, x_{t+1}, v_{t+1})]\}$$

(8)
subject to the laws of motion. Similar logic gives us a scaled law of motion for \( \tilde{k}_t \),

\[
g_{t+1}\tilde{k}_{t+1} = f(\tilde{k}_t, 1) - \tilde{c}_t. \tag{9}
\]

From here on, we drop the 1 and write the value function as \( J(\tilde{k}_t, x_t, v_t) \).

We find it convenient to work with the log of \( J \). With the constant elasticity time aggregator (2), we can rewrite (8) as

\[
\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} \rho^{-1} \log \left\{ (1 - \beta)\tilde{c}^\rho + \beta \mu_t [g_{t+1} J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]^\rho \right\}. \tag{10}
\]

In the limiting case of \( \rho = 0 \) — and intertemporal elasticity of substitution \( \sigma = 1/(1 - \rho) = 1 \) — we have

\[
\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c} + \beta \log \mu_t [g_{t+1} J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})].
\]

Additive models generally work with \( J^\rho / \rho t \).

\[
J(\tilde{k}_t, x_t, v_t)^\rho / \rho_t = \max_{\tilde{c}_t} (1 - \beta) \tilde{c}^\rho / \rho + \beta \mu_t [g_{t+1} J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]^\rho / \rho_t.
\]

If we redefine the value function as \( \tilde{J}_t = J^\rho / \rho_t \) and set \( \alpha = \rho \) (the additive case), the second term becomes \( \mu_t (g_{t+1} J_t)^\rho / \rho = E_t (g_{t+1}^\rho \tilde{J}_t) \), which is what we use in the additive case covered in Appendix C.

### 3.2 Asset pricing fundamentals

The planning problem generates a decision rule for consumption. The behavior of asset prices then follows from the representative agent’s intertemporal marginal rate of substitution. With recursive preferences, the marginal rate of substitution has the form

\[
m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho - 1} \left( \frac{U_{t+1}}{\mu_U (U_{t+1})} \right)^{\sigma - \rho}. \tag{11}
\]

See Appendix A. The first term is familiar from additive power utility. The next one is what we might term the Epstein-Zin term; it reflects the non-additivity of preferences over time and across states. It disappears if we set \( \rho = \alpha \), which puts us back in the additive case.

The marginal rate of substitution summarizes how the agent values assets with uncertain returns. In general no-arbitrage environments, there exists a positive pricing kernel \( m_{t+1} \) that satisfies

\[
E_t (m_{t+1} r_{t+1}) = 1
\]

for gross returns \( r_{t+1} \) on all assets. In representative agent models, the pricing kernel is the marginal rate of substitution and the equation is one of the agent’s first-order conditions.
The return on an asset depends on the cash flows to which it’s a claim, but we can get a sense of returns from the pricing kernel alone. The return $r_{t+1}$ with the highest log expectation, $E_t \log r_{t+1}$, is $r_{t+1} = 1/m_{t+1}$. And if $r^1_{t+1}$ is the gross return on a one-period (riskfree) bond, then the (logarithmic) risk premium on an arbitrary asset with return $r_{t+1}$ is $E_t(r_{t+1} - r^1_{t+1})$. The entropy bound tells us that risk premiums are bounded above by

$$E_t(\log r_{t+1} - \log r^1_{t+1}) \leq L_t(m_{t+1}),$$

(12)

where $L_t(m_{t+1}) = \log E_t(m_{t+1}) - E_t(\log m_{t+1})$ is the conditional entropy of the pricing kernel. See Backus, Chernov, and Zin (2014, Section I). The average maximum risk premium is the mean of this: $EL_t(m_{t+1})$.

4 Risk in the recursive Brock-Mirman example

Now that we have a handle on risk, we can explore its impact on the properties of business cycle models. We take a textbook standard and add a number of bells and whistles. None of the bells and whistles affect the decision rule for consumption, so you might ask why we bothered. We do it to illustrate some general properties in an example we can solve with pen and paper.

4.1 Model and solution

What’s come to be known as the Brock-Mirman example shows up in many introductions to dynamic programming for economists. In the notation of the previous section, it consists of additive preferences ($\rho = \alpha = 0$), log utility ($\rho = 0$), Cobb-Douglas technology ($f(a,k) = k^\omega a^{1-\omega}$ with $0 \leq \omega < 1$), and one hundred percent depreciation ($\delta = 1$). With this structure, the value function and decision rule are loglinear. See, for example, Appendix B. The same is true of the recursive version, where we allow the risk aversion parameter $\alpha$ to take on nonzero values in the certainty equivalent (3).

The Bellman equation for this recursive Brock-Mirman example is

$$\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c}_t + \beta \log \mu_t[g_{t+1}J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]$$

subject to (9), (6), and (7). Equation (9) in this case is $g_{t+1} \tilde{k}_{t+1} = f(\tilde{k}_t, 1) - \tilde{c}_t = \tilde{k}_t^\omega - \tilde{c}_t$.

When $\omega = 0$, we have the Bansal-Yaron (2004) asset pricing model with $\rho = 0$.

We find the solution by guess and verify. We guess the value function is loglinear:

$$\log J(\tilde{k}_t, x_t, v_t) = p_0 + p_k \log \tilde{k}_t + p_x x_t + p_v v_t$$
with coefficients \((p_0, p_k, p_x)\) to be determined. Next we substitute the laws of motion into next period’s value function and evaluate the certainty equivalent:

\[
\log[g_{t+1}J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})] = (1 - p_k)(\log g + e^\top x_{t+1}) + p_0 + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t) + p_x^\top x_{t+1} + p_v v_{t+1}
\]

\[
= p_0 + (1 - p_k) \log g + p_v(1 - \varphi)v + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t)
\]

\[
+ [(1 - p_k)e + p_x]^\top (Ax_t + Bv_t^{1/2}w_{t+1}) + p_v[\varphi v_t + \tau^{1/2}w_{2t+1}]
\]

\[
\log \mu_t[g_{t+1}J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})] = p_0 + (1 - p_k) \log g + p_v(1 - \varphi)v + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t)
\]

\[
+ [(1 - p_k)e + p_x]^\top Ax_t + [\alpha V_x/2 + p_v \varphi] v_t + (\alpha/2)(p_v)^2 \tau,
\]

where \(V_x = [(1 - p_k)e + p_x]^\top BB^\top [(1 - p_k)e + p_x]\). Note the risk adjustment \(\alpha V_x/2\) in the certainty equivalent, it’s central to how risk works in such models and shows how the risk parameter \(\alpha\) magnifies its effect.

If we substitute the certainty equivalent into the Bellman equation, the first order condition,

\[
(1 - \beta)/\tilde{c}_t = \beta p_k/(\tilde{k}_t^\omega - \tilde{c}_t),
\]

gives us the decision rule

\[
\tilde{c}_t = \{(1 - \beta)/[\beta p_k + (1 - \beta)]\} \tilde{k}_t^\omega.
\]

Traditionally we would substitute this back into the Bellman equation and solve for \(p_k\). Here there’s a simpler method. The envelope condition for \(\tilde{k}_t\) gives us the decision rule

\[
\tilde{c}_t = (1 - \beta \omega) \tilde{k}_t^\omega.
\]

The controlled law of motion is therefore \(g_{t+1}\tilde{k}_{t+1} = \tilde{k}_t^\omega - \tilde{c}_t = \beta \omega \tilde{k}_t^\omega\). If we were searching for a loglinear decision rule, something like

\[
\log \tilde{c}_t = h_{c0} + h_{ck} \log \tilde{k}_t + h_{cx}^\top x_t + h_{cv} v_t,
\]

then we’ve found it: \(h_{c0} = \log(1 - \beta \omega)\) and \(h_{cv} = \omega\). Note that \(h_{cx} = h_{cv} = 0\): so neither news \(x_t\) nor uncertainty \(v_t\) affects scaled consumption.

If we substitute the decision rule into the Bellman equation and line up terms, we find

\[
p_k = (1 - \beta) \omega/(1 - \beta \omega)
\]

\[
p_x^\top = \beta(1 - p_k)e^\top A(I - \beta A)^{-1}
\]

\[
p_v = \beta(\alpha/2)V_x/(1 - \beta \varphi)
\]

\[
(1 - \beta)p_0 = (1 - \beta) \log(1 - \beta \omega) + \beta(1 - p_k) \log g + \beta p_v(1 - \varphi)v + \beta p_k \log(\beta \omega) + \beta \alpha(p_v)^2 \tau/2.
\]

The coefficient \(p_k\) is between zero and one, which makes \(J(\tilde{k}_t, x_t, v_t)\) increasing and concave in \(\tilde{k}_t\). The coefficient \(p_x\) captures the predictability of log productivity growth: if \(A = 0\), so that productivity growth is unpredictable, then \(p_x = 0\) as well. And if \(\alpha < 0\), as we’ll typically assume, then \(p_v < 0\): an increase in \(v_t\) lowers utility.
4.2 Properties

This problem has a number of features of general interest. Among them:

- **Tallarini property.** Quantities don’t depend on risk or risk aversion. This is an illustration of Tallarini’s (2000) result: to a loglinear approximation, the decision rules, and therefore the properties of quantities, are approximately the same with additive ($\alpha = \rho$) and recursive (arbitrary $\alpha$) preferences. Here the result is exact. The risk aversion parameter $\alpha$ affects the value function (see $p_0$), but not the decision rule for consumption.

- **Asset prices do.** This is in Tallarini (2000), too: asset prices and risk premiums depend on $\alpha$. Here’s an example. The pricing kernel has these components:

\[
\log(\tilde{c}_{t+1}/\tilde{c}_t) = \omega \log(\beta\omega) + \omega(\omega - 1) \log \tilde{k}_t - \omega \log g_{t+1}
\]

\[
\log(c_{t+1}/c_t) = \log(\tilde{c}_{t+1}/\tilde{c}_t) + \log g_{t+1}
\]

\[
= \omega \log(\beta\omega) + (1 - \omega) \log g + \omega(\omega - 1) \log \tilde{k}_t + (1 - \omega)e^\top Ax + (1 - \omega)e^\top Bv_1^{1/2} w_{1t+1}
\]

\[
\log[g_{t+1}J_{t+1}/\mu_t(g_{t+1}J_{t+1})] = [(1 - p_k)e + p_x]^\top Bv_1^{1/2} w_{1t+1} + p_v \tau^{1/2} w_{2t+1}
\]

\[- (\alpha/2) V_m v_t - (\alpha/2)(p_v)^2 \tau.
\]

The pricing kernel is therefore

\[
\log m_{t+1} = \log \beta - \log(c_{t+1}/c_t) + \alpha \log[g_{t+1}J_{t+1}/\mu_t(g_{t+1}J_{t+1})]
\]

\[
= \log \beta - \omega \log(\beta\omega) - (1 - \omega) \log g - \alpha(\alpha/2)(p_v)^2 \tau
\]

\[- \omega(\omega - 1) \log \tilde{k}_t - (1 - \omega)e^\top Ax - \alpha(\alpha/2)V_x v_t
\]

\[+ \{\alpha[(1 - p_k)e + p_x] - (1 - \omega)e\}^\top Bv_1^{1/2} w_{1t+1} + \alpha p_v \tau^{1/2} w_{2t+1}.
\]

Conditional on the state at date $t$, most of this is constant. All the variation, and therefore all risk premiums, come from the last two terms.

The entropy bound (12) gives us the maximum risk premium,

\[
L_t(m_{t,t+1}) = (1/2)[V_m v_t + (\alpha p_v)^2 \tau],
\]

with

\[
V_m = \{\alpha[(1 - p_k)e + p_x] + (1 - \omega)e\}^\top BB^\top\{\alpha[(1 - p_k)e + p_x] - (1 - \omega)e\}.
\]

Both terms are affected by $\alpha$. In a typical case, increasing risk aversion (meaning larger negative values of $\alpha$) increases the risk premium. This mirrors Tallarini, where risk premiums are affected by risk aversion, but consumption is not.

- **Separation property.** This model, like most dynamic programs in economics, includes both endogenous ($\tilde{k}_t$) and exogenous ($x_t, v_t$) state variables. Anderson, Hansen, McGrattan, and Sargent (1996, Sections 2 and 3) show that in analogous linear-quadratic control
problems, the components of the solution separate: the coefficients of the endogenous state variables in the value function and decision rules do not depend on the properties of the exogenous state variables and can be computed separately.

This problem has a similar feature. The coefficients \( p_k = (1 - \beta)\omega/(1 - \beta \omega) \) in the value function and \( h_{ck} = \omega \) in the decision rule do not depend on the parameters governing the dynamics of \( (x_t, v_t) \). We illustrate this feature by plotting \( \log \tilde{k}_{t+1} \) against \( \log \tilde{k}_t \) for given values of \( (g_{t+1}, v_t, x_t) \). The separation property tells us that the slope of this line is the same for all values of these other variables. Here the line takes the form

\[
\log \tilde{k}_{t+1} = \log(\beta \omega) + \omega \log \tilde{k}_t - \log g_{t+1}.
\]

We see that the slope of the line, namely \( \omega \), does not depend on the state variable \( v_t \) or its properties.

5 Risk in the recursive business cycle model

The Brock-Mirman example is illustrative, but its simplicity is misleading. If we change the technology or allow the IES to differ from one, the model isn’t nearly as tractable. It is, however, solvable by loglinear approximation methods not much different from Campbell’s (1994). It can also be solved, of course, by any number of numerical methods, but the approximations have the advantage of transparency: we can see exactly how they work and which features determine their properties.

5.1 Model and solution

The problem consists of the Bellman equation (10), the certainty equivalent (3), and the laws of motion (9), (6), and (7). The first-order and envelope conditions are

\[
0 = J_t^{\rho} \{(1 - \beta)\tilde{c}_t^{\rho-1} - \beta \mu_t (g_{t+1} J_{t+1})^{\rho-\alpha} E_t[(g_{t+1} J_{t+1})^{\alpha-1} J_{kt+1}]\}
\]

\[
J_{kt}/J_t = J_t^{\rho-\alpha} \beta \mu_t (g_{t+1} J_{t+1})^{\rho-\alpha} E_t[(g_{t+1} J_{t+1})^{\alpha-1} J_{kt+1}] f_{kt}.
\]

Together they imply

\[
(1 - \beta)\tilde{c}_t^{\rho-1} = J_t^{\rho-1} J_{kt}/f_{kt}.
\]

This is similar to what we’d get in the additive model; see Appendix C. The resemblance is closer if we transform the value function: \( \tilde{J}_t = J_t^\rho / \rho \). Then the derivative of the value function is \( \tilde{J}_{kt} = J_t^{\rho-1} J_{kt} \). With this transformation, we see that the decision rule — the solution of (15) — depends on the derivative of the value function but not the value function itself. That’s a general feature of additive dynamic programs with continuous control variables. It’s not true of the recursive model, where the first-order and envelope conditions involve the value function as well as its derivative.

The idea behind loglinearization is simple in the extreme: we take functions that are not loglinear and nevertheless approximate them by loglinear functions. We’ll see that this
works amazingly well for models of this kind. The approximations involve derivatives at a point, which for us is the stochastic steady state. [More on this another time.]

As an example, consider an arbitrary positive function \( f \) of a positive random variable \( x_t \). A linear approximation in logs around the point \( x_t = x \) is

\[
\log f(x_t) = \log f + (f_x x_f) (\log x_t - \log x).
\]

Typically we ignore the intercept and write this as \( \log f(x_t) = (f_x x_f) \log x_t \). Similarly, with two variables we have \( \log f(x_t, y_t) = (f_x x_f) \log x_t + (f_y y_f) \log y_t \). An example we use repeatedly is the marginal product of capital \( f_{kt} = f_k(\tilde{k}_t, 1) \), which we approximate by

\[
\log f_{kt} = (f_{kk} \tilde{k}/f_k) \log \tilde{k}_t = \lambda_k \log \tilde{k}_t.
\]

This is the gross return at date \( t \) on one unit of capital invested at \( t - 1 \). Another is the law of motion (4), which we approximate by \( \log \tilde{k}_{t+1} = \lambda_k \log \tilde{k}_t - \lambda_c \log \tilde{c}_t - e^\top x_{t+1} \).

The notation and approach are adapted from Campbell (1994).

Applying similar methods to dynamic programming problems, we derive loglinear approximations to the consumption decision rule for business cycle models. We derive Campbell's loglinear approximation for the additive case from a loglinear approximation to the derivative of the value function; see Appendix C. And in Appendix D we derive an analogous approximation for the recursive case. The result is a decision rule of the form (13) for consumption and a similar approximation to the controlled law of motion,

\[
\log \tilde{k}_{t+1} = h_{k0} + h_{kk} \log \tilde{k}_t + h_{kx} x_t + h_{kv} v_t.
\]

The approximation has the following general properties:

- Tallarini property. Without variation in risk — that is, with \( \tau = 0 \) in equation (7) — the loglinear approximations of the consumption decision rule are identical in the additive and recursive models. Recursive preferences are irrelevant here to the behavior of quantities, if not to asset prices.

- Separation property. More precisely: (i) the coefficients \( h_{ck} \) in equation (13) and \( h_{kk} \) in equation (18) do not depend on the behavior of either news \( x_t \) or risk \( v_t \) and (ii) the coefficient vector \( h_{cx} \) does not depend on risk.

- Accuracy. The loglinear approximation isn’t perfect, but if we plot \( \log \tilde{k}_{t+1} \) against \( \log \tilde{k}_t \) over the range of values generated by the model, it’s impossible to distinguish the solution produced by our numerical procedure from the loglinear approximation of (18). See Figure 5. A closer look (not reported) shows that the line isn’t precisely a straight line, but deviations from linearity are small. The same is true of the Tallarini and separation properties: they’re not exact, but they’re pretty good approximations.
- Risk. The coefficient $h_{cv}$ of risk in the consumption decision rule depends in a complicated way on both intertemporal substitution and risk aversion. The sign and magnitude both depend on parameter values, but for the choices we make below an increase in risk produces a decline in consumption. For a given level of productivity, output is fixed, so investment rises. This is an example of the Barro-King problem we mentioned earlier.

5.2 Properties

We can get a sense of the magnitude of the impact of risk if we choose specific parameter values. We compute numerical examples, rather than carefully calibrated models, but we’ve made some attempt to choose quasi-realistic parameter values, or at least parameter values with some history in the literature. Our choices are summarized in Table 1, but here’s the logic:

- Preferences. We set $\rho = -1$, which corresponds to an IES of 1/2.
- Technology: We use a Cobb-Douglas production function with capital share of 1/3. This corresponds to equation (5) with $\nu = 0$ and $\omega = 1/3$.
- Productivity growth. Our starting point is a random walk with stochastic volatility:
  \[ \log g_{t+1} = \log g + v_t^{1/2} w_{t+1}, \]
  which corresponds to setting $A = 0$ and $c = B = 1$.
- Technology: law of motion. We use Kydland and Prescott’s $\delta = 0.025$.
  (Talk about steady state capital-output ratio.)

With these inputs, we can compute numerical values for the various expressions used to compute the solution. The marginal product of capital is $f_k = s_k(y/k) + 1 - \delta$. The second derivative term
  \[ f_{kk} = -(1 - \nu)\omega(y/k)^{-\nu} [1 - \omega(y/k)^{-\nu}] = -(1 - \nu)(1 - s_k)s_k(y/k) = -0.0222. \]

The steady state investment-to-capital ratio follows from equation (4):
  \[ x_1 = 1 - \delta + (i/k), \]
  or $i/k = x_1 - 1 + \delta = 0.0290$. Then $(c/k) = (y/k) - (i/k) = 0.0710$ and $\lambda_c = c/kg = 0.0707$. [This needs work.]

We see the results in Figure 6 and Table 2. The figure illustrates the responses of (the logarithms of) capital and consumption to a unit increase in the innovation $w_{2t}$ in the conditional variance $v_t$. That’s a relatively small effect. A two standard deviation (of $v_t$) increase would be roughly six times larger. We see that the unit increase reduces consumption by a small amount in the benchmark case; roughly 0.02. If we increase risk
aversion to $1 - \alpha = 50$, this rises to 0.1. Multiplying by six, we would get a little more than a half percent decrease in consumption from a two standard deviation increase in risk. This is a noticeable effect, but it’s not hard to see that the model needs help to generate an impact as large as we saw in the Great Recession.

The statistics summarized in the table also suggest that the impact of uncertainty fluctuations is small in this model. The first column of the table includes a summary of the evidence taken from Tallarini (2000, Table 6). The statistics are based on (continuously-compounded) growth rates rather than some kind of filtered object, but the properties are familiar: the standard deviation of consumption is smaller, and the standard deviation of investment larger, than that of output. And both consumption and investment are positively correlated with output, although the correlations are smaller than we would see with (say) Hodrick-Prescott filtered variables.

The properties of the model are broadly similar, although the differences in standard deviations are smaller and the correlations are larger. More relevant to us is the role of uncertainty shocks. Column (3) summarizes the model with benchmark parameter values. Column (2) shows us that it’s not much different from the additive case. Column (5) shows how it changes when we eliminate variation in risk. The answer: hardly at all. We get a larger difference when we increase risk aversion to $1 - \alpha = 50$; column (4). This increases, as we have seen, the magnitude of the impact of fluctuations in uncertainty, but even in this case the impact on these statistics is modest. We have a bigger impact when we change the IES, as we do in column (6), but uncertainty shocks have little to do with this.

6 Ambiguity

We use the term ambiguity to describe situations in which the decision maker does not know the some aspect of the distribution of outcomes. Whether it’s a parameter or a state variable is in large part a matter of language. The critical aspect of ambiguity is that it’s treated differently by preferences than risk. Our treatment of ambiguity is built on the smooth ambiguity foundation laid by Jahan-Parvar and Liu (2012), Ju and Miao (2012), and especially Klibanoff, Marinacci, and Mukerji (2005, 2009). We find it more user-friendly than more popular approaches built on maxmin expected utility.

6.1 Smooth ambiguity

Consider ambiguity in a static setting with two sources of uncertainty, $s = (s_1, s_2)$. Consumption outcomes are defined over them by $c(s) = c(s_1, s_2)$. The sources of uncertainty are, first, the distribution of $s_1$ conditional on $s_2$ and, second, the distribution over $s_2$. There’s no difference between $s_1$ and $s_2$ at this level of generality, but in applications $s_2$ is often a parameter or a hidden state.

We denote the expectations based on these distributions by $E_1$ and $E_2$, respectively, and the overall expectation by $E = E_2 E_1$. More explicitly, the various expectations of an arbitrary
The certainty equivalent has two parts, 

\[ \mu(c) = \mu_2[\mu_1(c)], \]

where

\[ \mu_1(c) = u^{-1}[E_1u(c)], \]
\[ \mu_2[\mu_1(c)] = v^{-1}(E_2v[\mu_1(c)]). \]

These functions exhibit risk aversion if \( u \) is concave and ambiguity aversion if \( v \circ u^{-1} \) is concave — roughly speaking, if \( v \) is more concave than \( u \). The power utility versions are

\[ \mu_1(c) = \left( E_1(c^\alpha) \right)^{1/\alpha}, \quad \mu_2[\mu_1(c)] = \left( E_2[\mu_1(c)^\gamma] \right)^{1/\gamma} \]

with parameters \( \alpha < 1 \) and \( \gamma < \alpha \). Here \( 1 - \alpha \) is a measure of risk aversion and \( 1 - (\gamma - \alpha) \) is a measure of ambiguity aversion. If \( \gamma = \alpha \), this reduces to expected utility:

\[ \mu(c) = \mu_2[\mu_1(c)] = \left[ E_2E_1(c^\alpha) \right]^{1/\alpha} = \left[ E(c^\alpha) \right]^{1/\alpha}. \]

Alternatively, if we drive \( \gamma \) to minus infinity we get the popular maxmin expected utility. For values of \( \gamma \) between minus infinity and \( \alpha \), the smooth ambiguity model captures the idea of model uncertainty in a user-friendly way.

**Example (continued).** We illustrate the impact of ambiguity aversion in two variants of our earlier example. (i) Ambiguous mean. We express ambiguity over the mean with a two-part distribution. Part 1: Conditional on \( s_2 \), \( \log c = s_1 \sim \mathcal{N}(s_2, \kappa_2) \). Part 2: \( s_2 \sim \mathcal{N}(\kappa_1, \nu) \). The first certainty equivalent is \( \log \mu_1(c) = s_2 + \alpha \kappa_2/2 \). The overall certainty equivalent is \( \log \mu(c) = \log \mu_2[\mu_1(c)] = \kappa_1 + \gamma \nu/2 + \alpha \kappa_2/2 \). The mean satisfies \( \log E(c) = \kappa_1 + \kappa_2 + \nu/2 \), so the adjustment for risk is \( (1 - \alpha)\kappa_2/2 \) and the adjustment for ambiguity is \( (1 - \gamma)\nu/2 \). (ii) Ambiguous variance. Part 1: conditional on \( s_2 \), \( \log c = s_1 \sim \mathcal{N}(\kappa_1, \kappa_2) \). Part 2: \( s_2 \sim \mathcal{N}(\nu_1, \nu_2) \). The variance is normal and therefore negative with positive probability, which is impossible but analytically convenient. The certainty equivalents are \( \log \mu_1(c) = \kappa_1 + \alpha s_2/2 \) and \( \log \mu_2[\mu_1(c)] = \kappa_1 + \alpha \nu_1/2 + \gamma(\alpha/2)^2 \nu_2/2 \). Evidently there’s no clean separation here between the adjustments for risk and ambiguity.

Imbed in recursive structure, with the same time aggregator (2) and certainty equivalent

\[ \mu_t(U_{t+1}) = \mu_{2t}[\mu_{1t}(U_{t+1})]. \]

One byproduct is an extra term in the marginal rate of substitution:

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} \left( \frac{U_{t+1}}{\mu_{2t}[\mu_{1t}(U_{t+1})]} \right)^{\alpha - \rho} \left( \frac{\mu_{1t}(U_{t+1})}{\mu_{2t}[\mu_{1t}(U_{t+1})]} \right)^{\gamma - \alpha}. \]

(20)
6.2 Two examples

1. Ambiguity about the mean with constant signal variance

2. Stochastic signal variance

6.3 Properties

Do some numerical examples...

7 Related work

Related work...


Ambiguity. As we noted, our approach to ambiguity follows Jahan-Parvar and Liu (2012), Ju and Miao (2012), and Klibanoff, Marinacci, and Mukerji (2005, 2009). The smooth ambiguity in these papers is much closer to the traditional treatment of risk and brings out the similarities between risk and ambiguity. Collard, Mukerji, Sheppard, and Tallon (2012) discuss how one might identify their effects.

Computation. We’ve noted the connection to Campbell (1994), which is easily adapted to incorporate uncertainty. Somewhat different is a large body of work based on “perturbation methods.” Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Wen (2012) is a good example. These methods start with a first-order approximation of models with all the uncertainty turned off. Risk enters only in higher-order approximations. It is, evidently, possible to bring in uncertainty even with first-order approximations, as a large literature on exponential-affine asset pricing models attests. We’ve done the same here.

Endogenous risk. The key to the separation property is that there’s no feedback from the economy to uncertainty. Fajgelbaum, Schaal, and Taschereau-Dumouchel (2014) and Veldkamp (2005) are striking examples of how internal dynamics change once uncertainty responds to the state of the economy.

8 Last thoughts

We have described how risk and ambiguity might affect the dynamics of quantities in a traditional business cycle model. The short answer: The impact is small. A longer answer:
Variations in uncertainty in this model have, for reasonable parameter values, a small impact on consumption, investment, and output. They have no impact on the internal dynamics of the capital stock: on the speed at which the economy recovers from a temporary increase in uncertainty. Prolonged uncertainty could, of course, generate a prolonged recession on its own, but the evidence suggests that this hasn’t been the case in the US.

Which leads us to ask: What mechanisms might lead to larger, more prolonged effects? One is to have uncertainty arise endogenously in response to economic events. Certainly we see that in the world around us. Such a mechanism would break the separation property and could lead to a role of uncertainty in the speed of a recovery. Fajgelbaum, Schaal, and Taschereau-Dumouchel (2014) and Veldkamp (2005) give striking examples of this.

Another mechanism is to have uncertainty affect individual firms or households. A good example is Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012). Other mechanisms might include financial frictions, such as those in Arellano, Bai, and Kehoe (2012) and Cooley, Marimon, and Quadrini (2004). If you have other ideas, please pass them on.
A Marginal rates of substitution

We derive the marginal rates of substitution for recursive preferences with risk and ambiguity. Consider an event tree with histories or states \( s^t = (s_0, s_1, \ldots, s_t) \). We’re interested in the marginal rate of substitution between consumption in state \( s^t \) and a succeeding state \( s^{t+1} = (s', s) \). Since everything starts at \( s^t \), we can ignore it in what follows. We use a finite state-space to simplify the notation.

Preferences are characterized by the time aggregator (2) and the risk and ambiguity certainty equivalents (19). Denote current utility by \( U(s^t) = U_t \) and future utility by \( U(s^t, s) = U_{t+1}(s) = U_t + 1 \). We divide \( s \) into \( (s_1, s_2) \) and consider probabilities \( \pi(s_1, s_2) = \pi_1(s_1 | s_2) \pi_2(s_2) \), all conditional on the current state \( s^t \). The overall certainty equivalent is \( \mu_t(U_{t+1}) = \mu_2[\mu_1(U_{t+1})] \). The inner one,

\[
\mu_1(U_{t+1}) = \left[ \sum_{s_1} \pi_1(s_1 | s_2) U_{t+1}(s_1, s_2)^\alpha \right]^{1/\alpha},
\]

might be expressed \( \mu_1(s_2) \), a function of \( s_2 \). The outer one is

\[
\mu_2[\mu_1(s_2)] = \left[ \sum_{s_2} \pi_2(s_2) \mu_1(s_2)^\gamma \right]^{1/\gamma}.
\]

Marginal utilities follow from repeated application of the chain rule. The relevant derivatives are

\[
\frac{\partial U_t}{\partial c_t} = U_t^{1-\rho} (1-\beta) c_t^{\rho-1},
\]

\[
\frac{\partial U_t}{\partial \mu_2[\mu_1(s_2)]} = U_t^{1-\rho} \beta \mu_2[\mu_1(s_2)]^{\rho-1},
\]

\[
\frac{\partial \mu_2[\mu_1(s_2)]}{\partial \mu_1(s_2)} = \mu_2[\mu_1(s_2)]^{1-\gamma} \pi_2(s_2) \mu_1(s_2)^{\gamma-1},
\]

\[
\frac{\partial \mu_1(s_2)}{\partial U_{t+1}(s_1, s_2)} = \mu_1(s_2)^{1-\alpha} \pi(s_1 | s_2) U_{t+1}(s_1, s_2)^{\alpha-1}.
\]

The marginal rate of substitution is therefore

\[
\frac{\partial U_t}{\partial c_t} \frac{\partial U_{t+1}(s_1, s_2)}{\partial c_t} = \frac{\partial U_t}{\partial \mu_2[\mu_1(s_2)]} \frac{\partial \mu_2[\mu_1(s_2)]}{\partial \mu_1(s_2)} \frac{\partial U_{t+1}(s_1, s_2)}{\partial c_t} \frac{\partial c_t}{\partial c_t} \left( \frac{c_t + 1}{c_t} \right)^{\rho-1} \left( \frac{U_{t+1}(s_1, s_2)^\alpha}{\mu_2[\mu_1(U_{t+1})]^{\gamma-\rho} \mu_1(U_{t+1})^{\alpha-\gamma}} \right) \\
= \pi(s_1 | s_2) \pi_2(s_2) \beta \left( \frac{c_t + 1}{c_t} \right)^{\rho-1} \left( \frac{U_{t+1}(s_1, s_2)^\alpha}{\mu_2[\mu_1(U_{t+1})]^{\gamma-\rho} \mu_1(U_{t+1})^{\alpha-\gamma}} \right).
\]

B The Brock-Mirman example

Ljungqvist and Sargent (2000, Chapter 4, Appendix B) give the traditional stripped-down version of the Brock-Mirman example. Agents have log utility, which corresponds to our
recursive preferences with \( \rho = \alpha = \gamma = 0 \). The laws of motion for capital and productivity are 
\[ k_{t+1} = y_t - c_t = a_t k_t^\omega - c_t \]
and 
\[ \log a_{t+1} = \varphi \log a_t + \tau^{1/2} w_{t+1} \]
where \( 0 \leq \omega < 1 \) and \( \{w_t\} \) is an iid sequence of standard normal random variables. The Bellman equation is
\[
\log J(k_t, a_t) = \max_{c_t} (1 - \beta) \log c_t + \beta E_t[\log J(k_{t+1}, a_{t+1})]
\]
subject to the laws of motion for \( k_t \) and \( a_t \).

We solve by guess and verify. We guess 
\[
\log J(k_t, a_t) = p_0 + p_k \log k_t + p_a \log a_t.
\]
Then next period’s value function and its expectation are
\[
\log J(k_{t+1}, a_{t+1}) = p_0 + p_k \log(a_t k_t^\omega - c_t) + p_a (\varphi \log a_t + \tau^{1/2} w_{t+1}),
\]
\[
E_t J_{t+1} = p_0 + p_k \log(a_t k_t^\omega - c_t) + p_a \varphi \log a_t.
\]
If we substitute into the Bellman equation, the envelope condition for \( k_t \) is
\[
p_k/k_t = \beta p_k \omega a_t k_t^\omega/(a_t k_t^\omega - c_t).
\]
That gives us the decision rule
\[
c_t = (1 - \beta \omega) a_t k_t^\omega,
\]
which implies the controlled law of motion \( k_{t+1} = \beta \omega a_t k_t^\omega \). The Bellman equation is then
\[
p_0 + p_k \log k_t + p_a \log a_t = (1 - \beta)[\log(1 - \beta \omega) + \log a_t + \omega \log k_t]
\]
\[
+ \beta \{p_0 + p_k [\log(\beta \omega) + \log a_t + \omega \log k_t] + p_a \varphi \log a_t\}.
\]
Lining up terms gives us the solution:
\[
p_k = (1 - \beta) \omega/(1 - \beta \omega) \Rightarrow 1 - p_k = (1 - \omega)/(1 - \beta \omega)
\]
\[
p_a = [(1 - \beta(1 - p_k))/(1 - \beta \omega) = (1 - \beta)/[(1 - \beta \omega)(1 - \beta \varphi)]
\]
\[
p_0 = [(1 - \beta) \log(1 - \beta \omega) + \beta p_k \log(\beta \omega)]/(1 - \beta).
\]
Note that \( 0 \leq p_k < 1 \), which makes \( J(k_t, a_t) \) (weakly) increasing and concave in \( k_t \).

C Approximations of the additive business cycle model

We compute an approximate loglinear solution to a business cycle model with additive preferences and constant volatility and compare it to Campbell’s (1994) solution of the same model. The approaches are different, but they deliver the same decision rule.

We approach the problem as a dynamic program. With additive preferences \( (\alpha = \rho) \) the Bellman equation can be written
\[
J(\tilde{k}_t, x_t) = \max_{\tilde{c}_t} (1 - \beta) \tilde{c}_t^\rho / \rho + \beta E_t [g_{t+1}^0 J(\tilde{k}_{t+1}, x_{t+1})].
\]
subject to the laws of motion \( \ddot{t}_{t+1} = [f(\ddot{t}_t, 1) - \dot{c}_t]/g_{t+1} \) and \( t_{t+1} = A_{xt} + B_{wt+1} \). Here we’ve taken equation (8) to the power \( \rho \), divided by \( \rho \), and redefined \( J_\rho/\rho \) as \( J_t \). The first-order and envelope conditions are

\[
(1 - \beta)\ddot{c}_t^{\rho-1} = \beta E_t(\gamma_{t+1} J_{kt+1})
\]

\[
J_{kt} = \beta E_t(\gamma_{t+1} J_{kt+1}) f_{kt}.
\]

Together they imply

\[
(1 - \beta)\ddot{c}_t^{\rho-1} = J_{kt}/f_{kt}.
\]

This illustrates a point we made earlier: in the additive case, we need the derivative of the value function but not the value function itself. Evidently a loglinear decision rule requires loglinear approximations of \( J_{kt} \) and \( f_{kt} \).

We guess the derivative of the value function has the form

\[
\log J_{kt} = q_k \log \dot{t}_t + q_x^T x_t
\]

with coefficients \((q_k, q_x)\) to be determined. Putting this together with the loglinear approximations (16, 17) gives us

\[
\log J_{kt} - \log f_{kt} = (q_k - \lambda_r) \log \dot{t}_t + q_x^T x_t
\]

\[
\log \ddot{c}_t = -\sigma(q_k - \lambda_r) \log \dot{t}_t - \sigma q_x^T x_t
\]

\[
\log \ddot{k}_{t+1} = [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] \log \dot{t}_t + \sigma \lambda_c q_x^T x_t - e^T x_{t+1}.
\]

Ordinarily we would substitute the decision rule for consumption into the Bellman equation and solve for \( q_k \). Here it’s sufficient to use the envelope condition, the derivative of the Bellman equation. The right-hand side involves

\[
\log(\gamma_{t+1} J_{kt+1}) = q_k \log \dot{t}_{t+1} + [q_x + (\rho - 1)\epsilon]^T x_{t+1}
\]

\[
= q_k [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] \log \dot{t}_t + q_k \lambda_c q_x^T x_t + [q_x + (\rho - 1 - q_k)\epsilon]^T (A_{xt} + B_{wt+1})
\]

\[
\log E_t(\gamma_{t+1} J_{kt+1}) = q_k [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] \log \dot{t}_t + \{ \sigma \lambda_c q_k q_x + [q_x + (\rho - 1 - q_k)\epsilon]^T A \} x_t + V_x/2,
\]

where \( V_x = [q_x + (\rho - 1)\epsilon]^T BB^T [q_x + (\rho - 1)\epsilon] \). The variance term \( V_x \) only shows up in the intercept, so we ignore it from here out. The envelope condition then gives us

\[
q_k \log \ddot{t}_t + q_x^T x_t = q_k [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] \log \dot{t}_t + \{ q_k \lambda_c q_k q_x + [q_x + (\rho - 1)\epsilon]^T A \} x_t + \lambda_r \log \ddot{k}_t.
\]

Equating similar terms, we have

\[
q_k = q_k [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] + \lambda_r
\]

\[
q_x^T = \sigma \lambda_c q_k q_x + [q_x + (\rho - 1)\epsilon]^T A.
\]
The first equation is quadratic in $q_k$ and has two solutions, one positive and one negative. We take the negative one, which corresponds to a concave value function and a stable controlled law of motion. Given solutions for $q_k$ and $q_x$, the decision rule for consumption follows immediately.

Campbell derives the same decision rule by another route. He starts with the “Euler equation,”

$$E_t[\beta(c_{t+1}/c_t)^{\rho-1} f_{kt+1}] = 1.$$  

Ignoring risk (which is constant, in any case), the loglinear version is

$$E_t(\log c_{t+1} - \log c_t) = -\sigma E_t(\log f_{kt+1}) = \sigma \lambda_r E_t(\log \tilde{k}_{t+1}).$$  \hspace{1cm} (21)

Where we use a guess for $J_{kt}$, he uses a guess for the consumption decision rule,

$$\log \tilde{c}_t = h_{ck} \log \tilde{k}_t + h_{cx}^T x_t,$$

with coefficients $(h_{ck}, h_{cx})$ to be determined. Using (16,17), the (controlled) law of motion for capital is

$$\log \tilde{k}_{t+1} = (\lambda_k - \lambda_c h_{ck}) \log \tilde{k}_t - \lambda_c h_{cx}^T x_t - e^T x_{t+1}.$$

The right side of the Euler equation (21) then becomes

$$\sigma \lambda_r E_t(\log \tilde{k}_{t+1}) = \sigma \lambda_r (\lambda_k - \lambda_c h_{ck}) \log \tilde{k}_t - \sigma \lambda_r (\lambda_c h_{cx}^T + e^T A) x_t.$$

The left side becomes

$$E_t(\log c_{t+1} - \log c_t) = h_{ck} (\log \tilde{k}_{t+1} - \log \tilde{k}_t) + h_{cx}^T (x_{t+1} - x_t) + e^T x_{t+1}$$

$$= [h_{ck} (\lambda_k - \lambda_c h_{ck}) - h_{ck}] \log \tilde{k}_t + [h_{cx}^T (A - I) + e^T A + h_{ck} \lambda_c h_{cx}^T] x_t.$$  

Equating the two gives us

$$h_{ck} (\lambda_k - \lambda_c h_{ck}) - h_{ck} = \sigma \lambda_r (\lambda_k - \lambda_c h_{ck})$$

$$h_{cx}^T (A - I) + e^T A + h_{ck} \lambda_c h_{cx}^T = -\sigma \lambda_r (\lambda_c h_{cx}^T + e^T A).$$

The first equation is Campbell’s equation (24) in slightly different notation. Kaltenbrunner and Lochstoer (2010) do the same. It's tedious but direct to show that the two solutions are identical: use the relation $h_{ck} = -\sigma(q_k - \lambda_r)$ to convert the quadratic in $h_{ck}$ into one for $q_k$.

**D Approximations of the recursive business cycle model**

We take two approaches to loglinear approximation of the model in Section 5. The first is based on a loglinear approximation to the Bellman equation suggested by Hansen, Heaton,
and Li (2008, Section III). The second is based on a loglinear approximation of the envelope condition, the derivative of the Bellman equation. We find the second more helpful, but in principle the two should give similar answers.

**Value function approximation.** Relative to the additive case of Appendix C, we need log-linear approximations of both the value function and its derivative. Consider a loglinear approximation to \( J_{kt} \):

\[
\log J_{kt} = p_1 + \log p_k + (p_k - 1) \log \tilde{k}_t + p_x^\top x_t + p_v v_t \\
\Rightarrow J_{kt} = p_k \tilde{k}_t^{p_k - 1} \exp(p_1 + p_x^\top x_t + p_v v_t). 
\]

If we integrate with respect to \( \tilde{k}_t \), we get the value function \( J_t = p_0 + \tilde{k}_t^{p_k} \exp(p_1 + p_x^\top x_t + p_v v_t) \)

It’s unfortunate that \( J_t \) isn’t loglinear unless \( p_0 = 0 \), but we use a loglinear approximation,

\[
\log J_t = d(p_k \log \tilde{k}_t + p_x^\top x_t + p_v v_t) 
\]

with \( d = (J - p_0)/J \). The combined term

\[
\log(J_t^{p_k - 1} J_{kt}) = \left\{ [1 + (\rho - 1)d]p_k - 1 \right\} \log \tilde{k}_t + [1 + (\rho - 1)d](p_x^\top x_t + p_v v_t).
\]

appears in the combined first-order and envelope condition (15).

**Hansen-Heaton-Li approximation.** The idea is to approximate (10), the log of the Bellman equation:

\[
\log J_t = \rho^{-1} \log \left[ (1 - \beta)e^{\rho \log \tilde{c}_t} + \beta e^{\rho \log \mu_t(g_{t+1} J_{t+1})} \right] \\
\cong b_0 + (1 - b_1) \log \tilde{c}_t + b_1 \log \mu_t(g_{t+1} J_{t+1}). \tag{22}
\]

This is exact if \( \rho = 0 \), in which case \( b_0 = 0 \) and \( b_1 = \beta \).

We need three things to put this to work: the value function \( J_t \), consumption \( \tilde{c}_t \), and the certainty equivalent of future utility \( \mu_t(g_{t+1} J_{t+1}) \). The first one we’ve done. Condition (15) then gives us the decision rule

\[
\log \tilde{c}_t = [(d - \sigma)p_k + \sigma(1 + \lambda_r)] \log \tilde{k}_t + (d - \sigma)(p_x^\top x_t + p_v v_t), 
\]

the second component of the approximate Bellman equation. The final component is the certainty equivalent of future utility. For that, we need the controlled law of motion (17),

\[
\begin{align*}
\log \tilde{k}_{t+1} & = \lambda_k \log \tilde{k}_t - \lambda_c \log \tilde{c}_t - e^\top x_{t+1} \\
& = \{ \lambda_k - \lambda_c[(d - \sigma)p_k + \sigma(1 + \lambda_r)] \} \log \tilde{k}_t - \lambda_c(d - \sigma)(p_x^\top x_t + p_v v_t) - e^\top x_{t+1}. 
\end{align*}
\]
Future utility and its certainty equivalent are then
\[
\log(g_{t+1}J_{t+1}) = \log g_{t+1} + d(p_k \log \tilde{k}_{t+1} + p_x^\top x_{t+1} + p_v v_{t+1})
\]
\[
= dp_k \{ \lambda_k - \lambda_c[(d - \sigma)p_k + \sigma(1 + \lambda_r)] \} \log \tilde{k}_t + \{(1 - dp_k)e^\top + dp_x^\top\}A - (\lambda_c dp_k(d - \sigma)p_x^\top) x_t + dp_v[\varphi_v - \lambda_c dp_k(d - \sigma)] v_t
\]
\[
+ [(1 - dp_k)e + dp_x] v_t^{1/2} B w_{t+1} + dp_v \tau w_{2t+1}
\]
\[
\log \mu_t(g_{t+1}J_{t+1}) = dp_k \{ \lambda_k - \lambda_c[(d - \sigma)p_k + \sigma(1 + \lambda_r)] \} \log \tilde{k}_t + \{(1 - dp_k)e^\top + dp_x^\top\}A - (\lambda_c dp_k(d - \sigma)p_x^\top) x_t + dp_v[\varphi_v - \lambda_c dp_k(d - \sigma)] + (\alpha/2)V_x v_t,
\]
where \( V_x = [dp_x + (1 - dp_k)e]^\top BB^\top[dp_x + (1 - dp_k)e] \) is the contribution of \( x_{t+1} \) to the conditional variance of \( \log(g_{t+1}J_{t+1}) \).

Now we plug this into the approximate Bellman equation and line up coefficients:
\[
p_k = (1 - b_1)[\sigma + (d - \sigma)p_k + \sigma \lambda_r] + b_1 dp_k \{ \lambda_k - \lambda_c[\sigma + (d - \sigma)p_k + \sigma \lambda_r] \}
\]
\[
p_x^\top = (1 - b_1)(d - \sigma)p_x + b_1 \{(1 - dp_k)e^\top + dp_x^\top\}A - \lambda_c dp_k(d - \sigma)p_x^\top
\]
\[
p_v = (1 - b_1)(d - \sigma)p_v + b_1 \{ dp_v[\varphi_v - \lambda_c dp_k(d - \sigma)] + (\alpha/2)V_x \} v_t.
\]
The first equation is quadratic in \( p_k \). Given a solution for \( p_k \), the equations for \( p_x \) and \( p_v \) are linear.

Envelope condition approximation. Our second method is based on the envelope condition and mirrors the approach we took to the additive model in Appendix C. The envelope condition for the recursive model is
\[
J_t^{\rho-1} J_{kt} = \beta \mu_t(g_{t+1}J_{t+1})^{\rho-\alpha} E_t[(g_{t+1}J_{t+1})^{\alpha-1}J_{kt+1}] f_{kt}.
\]
For the left side we reconstitute our earlier expression,
\[
\log(J_t^{\rho-1} J_{kt}) = \left\{ 1 + (\rho - 1)dp_k + 1 \right\} \log \tilde{k}_t + [1 + (\rho - 1)d](p_x^\top x_t + p_v v_t)
\]
\[
= q_k \log \tilde{k}_t + q_x^\top x_t + q_v v_t.
\]
The substitution of coefficients \((q_k, q_x, q_v)\) for \((p_k, p_x, p_v)\) is helpful because the same terms reappear elsewhere.

We need to evaluate \( g_{t+1}J_{t+1} \), which requires the decision rule and controlled law of motion. The decision rule in this notation is
\[
\log \tilde{c}_t = - \sigma(q_k - \lambda_r) \log \tilde{k}_t - \sigma(q_x^\top x_t + q_v v_t).
\]
That gives us the controlled law of motion
\[
\log \tilde{k}_{t+1} = [\lambda_k + \sigma \lambda_c(q_k - \lambda_r)] \log \tilde{k}_t + \sigma \lambda_c(q_x^\top x_t + q_v v_t) - e^\top x_{t+1}.
\]
Now back to the envelope condition (23). The critical components are

$$
\log(g_{t+1}J_{t+1}) = dp_k \log \tilde{k}_{t+1} + (e + dp_x)^\top x_{t+1} + dp_v v_{t+1} \\
= dp_k [\lambda_k + \sigma \lambda_c (q_k - \lambda_r)] \log \tilde{k}_t + dp_k \lambda_c q_x^\top x_t + dp_k \lambda_c \sigma q_v v_t \\
+ [(1 - dp_k) e + dp_x]^\top (Ax_t + v_t^{1/2} Bw_{1t+1}) + dp_v [(1 - \varphi_v) v + \varphi_v v_t + \tau w_{2t+1}]
$$

$$
\log J_{kt+1} = (p_k - 1) \log \tilde{k}_{t+1} + p_x^\top x_{t+1} + p_v v_{t+1} \\
= (p_k - 1) [\lambda_k + \sigma \lambda_c (q_k - \lambda_r)] \log \tilde{k}_t + (p_k - 1) \lambda_c q_x^\top x_t + (p_k - 1) \lambda_c \sigma q_v v_t \\
+ [(1 - p_k) e + p_x]^\top (Ax_t + v_t^{1/2} Bw_{1t+1}) + p_v [(1 - \varphi_v) v + \varphi_v v_t + \tau w_{2t+1}],
$$

which show up in $\mu_t(g_{t+1}J_{t+1})$ and $E_t[(g_{t+1}J_{t+1})^{\alpha-1} J_{kt+1}]$. We collect similar terms and find:

- Capital. These terms show up only in the mean. If we work through the right side of the envelope condition, we see that the mean terms are multiplied by $(\rho - \alpha) + (\alpha - 1) = \rho - 1$. The log $\tilde{k}_t$ terms are therefore

$$q_k = (\rho - 1) dp_k [\lambda_k + \sigma \lambda_c (q_k - \lambda_r)] + (p_k - 1) [\lambda_k + \sigma \lambda_c (q_k - \lambda_r)] + \lambda_r$$

In the first line, the first term on the right comes from $\log(g_{t+1}J_{t+1})$ and the second from $\log J_{kt+1}$. Through some piece of luck, they combine nicely. We are left with a quadratic in the coefficient $q_k$, in fact the same one we derived in Appendix C for the additive case.

As a result, the coefficients $h_{ck}$ of the approximate decision rule (13) and $h_{kk}$ of the controlled law of motion (18) are the same in the additive and recursive cases. As in Tallarini (2000), the risk aversion parameter plays no role in what we call the “internal dynamics” of the capital stock. Note, too, that the behavior of the exogenous state variables $x_t$ and $v_t$ have no impact on the solution. This is what we call the separation property: the capital coefficients $(q_{kk}, h_{ck}, h_{kk})$ are independent of the rest of the model.

- News. Again, $x_t$ shows up only in the mean terms, so we find the coefficient $q_x$ in much the same way. The relevant terms are

$$q_x^\top = q_k \lambda_c \sigma q_x^\top + [(\rho - 1) e + q_x] \lambda_r$$

This is, again, the same as the additive case we solved earlier. It’s also independent of risk ($v_t$) and risk aversion ($\alpha$): capital dynamics enter through $q_k$, but the properties of risk don’t affect the response to news. When $A = 0$, there’s no persistence in $x_t$ and $q_x = 0$.

- Risk. This one’s more involved, it incorporates risk and recursive preferences in a fundamental way. The coefficients of $v_t$ might be collected in two groups:

$$q_v = \text{mean terms} + \text{variance terms}.$$

24
The former are similar to what we’ve done:

\[
\text{mean terms} = (q_k \lambda c \sigma) q_0.
\]

The latter involve the variances of terms containing \(v_t^{1/2}\) in the envelope condition (23) in both the certainty equivalent \(\mu_t(g_{t+1}J_{t+1})\) and the expectation \(E_t[(g_{t+1}J_{t+1})^{\alpha-1}J_{kt+1}]\). Adding them in order, we have

\[
\text{variance terms} = (\rho - \alpha)\alpha V_1/2 + V_2/2
\]

where

\[
V_1 = [(1 - dp_k)e + dp_x]^\top BB^\top[(1 - dp_k)e + dp_x]
\]

\[
V_2 = \{(\alpha - 1)[(1 - dp_k)e + dp_x] + [(1 - p_k)e + px]\}^\top BB^\top
\]

\[
\{\{\alpha - 1][(1 - dp_k)e + dp_x] + [(1 - p_k)e + px]\}^\top BB^\top
\]

We need a value for \(d\) to implement this, but we can get a sense of where the sign comes from. Both \(V_1\) and \(V_2\) are positive, so the sign depends on their relative magnitudes and the sign of \(\rho - \alpha\).

**E  A business cycle perspective on Bansal and Yaron**

|later|

**F  Computational methods**

We used two methods besides loglinear approximation to verify the numbers reported in the paper. One uses a piecewise linear approximation to the value function. The other replaces the continuous state variables with discrete grids.

[more to come]
References


Notes: The figure plots the last five recessions starting from the previous peak. From Bethune, Cooley, and Rupert (2014).
Figure 2
Consumption over the last five recessions

Notes: The figure plots the last five recessions starting from the previous peak. From Bethune, Cooley, and Rupert (2014).
Figure 3
Nonresidential investment over the last five recessions

Notes: The figure plots the last five recessions starting from the previous peak. From Bethune, Cooley, and Rupert (2014).
Figure 4
Labor productivity over the last five recessions

Notes: The figure plots the last five recessions starting from the previous peak. From Bethune, Cooley, and Rupert (2014).
Figure 5
Slope of controlled law of motion: loglinear approximation and numerical solution
Figure 6
Response to a one standard deviation increase in conditional variance
Table 1
Benchmark parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Preferences</td>
<td></td>
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</tr>
<tr>
<td>$\rho$</td>
<td>$-1$</td>
<td>$\sigma = 1/2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-9$</td>
<td>Bansal &amp; Yaron (2004, Table II)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>—</td>
<td>chosen to hit $k/y = 10$ (quarterly)</td>
</tr>
<tr>
<td>(b) Technology</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0</td>
<td>Cobb-Douglas</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$1/3$</td>
<td>Kydland &amp; Prescott (1982, Table I), rounded off</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>Kydland &amp; Prescott (1982, Table I)</td>
</tr>
<tr>
<td>(c) Productivity growth</td>
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<td></td>
</tr>
<tr>
<td>$\log g$</td>
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<td>Tallarini (2000, Table 4)</td>
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<tr>
<td>$e$</td>
<td>1</td>
<td>normalization</td>
</tr>
<tr>
<td>$A$</td>
<td>0</td>
<td>no predictable component (news)</td>
</tr>
<tr>
<td>$B$</td>
<td>1</td>
<td>normalization</td>
</tr>
<tr>
<td>$v^{1/2}$</td>
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<td>Tallarini (2000, Table 4), rounded off</td>
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<tr>
<td>$\varphi_v$</td>
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</tr>
<tr>
<td>$\tau$</td>
<td>$0.74 \times 10^{-5}$</td>
<td>makes $v$ three standard deviations from zero</td>
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### Table 2
Properties of business cycle models with risk

<table>
<thead>
<tr>
<th>Variable</th>
<th>US Data (1)</th>
<th>Additive (2)</th>
<th>Recursive (3)</th>
<th>Recursive (4)</th>
<th>Recursive (5)</th>
<th>Recursive (6)</th>
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</thead>
<tbody>
<tr>
<td><strong>(a) Parameters</strong></td>
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<td>Changes from benchmark</td>
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<td><strong>(b) Standard deviations (%)</strong></td>
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<td>0.75</td>
<td>0.76</td>
<td>0.75</td>
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<td>1.04</td>
<td>1.06</td>
<td>1.02</td>
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<tr>
<td><strong>(c) Standard deviations relative to output growth</strong></td>
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<tr>
<td>Consumption growth</td>
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<td>0.91</td>
<td>0.91</td>
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<td>Investment growth</td>
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<td>1.27</td>
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<td>1.24</td>
<td>1.24</td>
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<td><strong>(d) Correlations with output growth</strong></td>
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<td>0.97</td>
<td>0.99</td>
<td>0.99</td>
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<tr>
<td>Investment growth</td>
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<td>0.97</td>
<td>0.93</td>
<td>0.98</td>
<td>0.98</td>
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</table>

Notes. US data from Tallarini (2000, Table 6).