

Tax distortions and the case for price stability *

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Abstract

We investigate the case for price stability in the general version of the New Keynesian (NNS) model with capital and several shocks. The model includes, in addition to the standard imperfect competition and monetary frictions, a nontrivial, endogenous tax distortion. We find that even under this specification the case for perfect price stability is not significantly weakened. Optimal policy tolerates a small amount of output gap and price variability by reacting less strongly to supply and fiscal shocks in comparison to a policy that aims at perfect price stabilization.

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Introduction

The optimality of price stability has been extensively investigated within the New Neoclassical Synthesis, NNS (or, new Keynesian, NK) model. A key result obtained is that perfect price stability is optimal when price rigidity is the only distortion present in the model. In this case, the flexible price equilibrium is efficient and thus optimal to replicate by stabilizing prices.

The situation is more complicated when the economy under consideration includes additional distortions. For instance, when the imperfect competition friction is not eliminated via a fiscal subsidy and/or money serves an explicit purpose in the model.

The presence of a monetary friction induces an incentive for sustained deflation (Friedman's zero interest rate rule). Furthermore, it may require systematic variation of the price level with the various shocks in order to stabilize the nominal interest rate and real balances. Both of these considerations suggest that some deviations from perfect price stabilization may be optimal. In practice, such deviations are likely to be small, because money typically matters little for economic activity and welfare in the flexible price equilibrium (see Woodford (2003), ch. 6).

When the imperfect competition friction is the only distortion besides price rigidity, Goodfriend and King (2001) show that perfect price stability remains the optimal policy as long as the elasticity of labor is constant, investment is absent and the only shocks are supply ones. They also conjecture that optimal departures from price stability are likely to be small even in the presence of fiscal shocks, of variable labor elasticity and of capital accumulation.

A similar result emerges when both of these distortions are present, but capital is absent. Khan, King and Wolman (2003) show that while optimal policy may involve some sustained deflation, optimal *price variation* in response to various shocks is likely to be negligible.

A more general case involves the co-existence of the monetary and imperfect competition distortions, capital accumulation and a variety of shocks. Nonetheless, Collard and Dellas (2003) find that even under these circumstances, the case for perfect price stabilization is not significantly weakened. *Some* inflation variability may prove welfare improving when price flexibility is large enough (a contract length less of one and a half quarter) and capital adjustment costs are very low.

Collard and Dellas argue that what seems to lie behind this result is the fact that the

wedge between the natural rate and the efficient level of output is either constant (with the monopolistic distortion) or small and little variable (with the monetary friction). With a small and little variable gap between the flexible price and the efficient equilibrium, the presence of investment does not induce a quantitatively significant deviation from perfect price stability (as conjectured by Goodfriend and King (2001), and Woodford (2003)).

Woodford (2003) has studied the properties of optimal policy in an economy with large and variable distortions. He establishes that strict price stabilization is no longer optimal. But he does not provide any indication of how quantitatively important the optimal deviations from perfect price stability are likely to be in practice. This is due to the fact that his approach relies on several convenient but restrictive features that are needed in order to allow the analytical derivation of the approximation to the welfare function and optimal policy (and also assumes that the source of variation in the inefficiency wedge—the mark up—is exogenous), so it is not suited to quantitative analysis.

The objective of this paper is to fill this gap. Namely, to investigate the degree of optimal deviation from perfect price stability as well as the properties of optimal policy in an economy that has a non-trivial, endogenous steady state distortion which also varies over the business cycle. The distortion under consideration is progressive income taxation. The model is solved using a second-order approximation as proposed by Collard and Juillard (2001) or Schmitt-Grohé and Uribe (2004).

We ask whether and to what extent the existence of variable tax distortions supports deviations from perfect price stability. The main results are the following: First, optimal monetary policy involves some tolerance of price instability. Second, optimal departures from perfect price stability are practically small. This obtains in spite of the fact that we use nominal price adjustment costs, rather than Calvo staggering. Use of the latter can, in some circumstances, exaggerate the cost of price dispersion. We speculate that this finding may be due to the fact that even a large, variable tax distortion does not necessarily generate sufficiently large variability in the inefficiency wedge (the difference between natural and efficient output). Third, optimal policy has two properties: a) it responds more weakly to supply and fiscal shocks in comparison to a policy that perfectly stabilizes the price level. And b) it attempts to keep actual fairly but not completely close to efficient output. And fourth, the standard interest policy rule specification used in the literature (the popular Henderson-McKibbin-Taylor rule) does not represent a good approximation of the optimal rule.

The remaining of the paper is as follows. Section 1 describes the model. Section 2

presents the calibration. Section 3 presents and discusses the main results. Section 4 offers some concluding remarks. Many of the derivations and the technical details appear in the appendix.

1 The Model

The set up is the standard NNS model. The economy is populated by a large number of identical infinitely-lived households and consists of two sectors: one producing intermediate goods and the other a final good. The intermediate good is produced with capital and labor and the final good with intermediate goods. The final good is homogeneous and can be used for consumption (private and public) and investment purposes.

1.1 The Household

Household preferences are characterized by the lifetime utility function:¹

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{(c_{t+j}^\nu \ell_{t+j}^{1-\nu})^{1-\sigma} - 1}{1-\sigma} \quad (1)$$

where c_t is consumption, ℓ_t is leisure, $0 < \beta < 1$ is a constant discount factor, $\sigma \in \mathbb{R}_+ \setminus \{1\}$ is the inverse of the intertemporal elasticity of substitution and $0 \leq \nu \leq 1$ is the weight attached to leisure in the utility index.

The household faces the following time constraint

$$\ell_t + h_t = 1 \quad (2)$$

where h denotes hours worked. The total time endowment is normalized to unity.

The household enters period t carrying money holdings M_t and nominal bonds D_t — which yield a nominal gross interest rate R_{t-1} — from the previous period. It receives a lump sum transfer, τ_t^m , from the monetary authorities and its share of profits, Π_t , from the firms. It supplies labor, h_t , at the real wage rate w_t and capital services, k_t , remunerated at the real rate of z_t . These funds are used to finance the purchase of money holdings, M_t , bonds, D_t , investment, $P_t x_t$ and consumption, $P_t(1 + \eta(v_t, \zeta_t))c_t$. $\eta(v_t; \zeta_t)$ is a proportional transaction cost that depends on the household's money-to-nominal consumption ratio

$$v_t = \frac{P_t c_t}{M_t}$$

¹ $E_t(\cdot)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period t .

The properties of $\eta(v_t; \zeta_t)$ are such (see Scmitt-Grohe and Uribe [2004] as well as the calibration section) that there is a satiation level of real balances, that the demand for money is decreasing in the nominal interest rate and that the transaction cost is zero when the nominal interest rate is zero.

Hence, the representative household faces the following budget constraint

$$D_t + M_t + P_t(1 + \eta(v_t, \zeta_t))c_t + P_t x_t + \leq (1 - \tau_t)P_t w_t h_t + (1 - \tau_t)P_t z_t k_t + R_{t-1}D_t + M_{t-1} + \Pi_t + \tau_t^m \quad (3)$$

where τ_t is a progressive tax rate, whose properties will be specified later.

Investment augments the capital stock according to

$$k_{t+1} = x_t - \frac{\varphi}{2} \left(\frac{x_t}{k_t} - \delta \right)^2 k_t + (1 - \delta)k_t \quad (4)$$

where $\delta \leq 0 \leq 1$ is the constant depreciation rate and $\varphi > 0$ is a parameter that determines the size of capital adjustment costs

The household determines its plans for consumption/savings, money holdings and leisure by maximizing utility (1) subject to the time (2) and budget (3) constraints and the law of motion of capital (4).

1.2 Final sector

The final good is produced by combining intermediate goods. This process is described by the following CES function

$$y_t = \left(\int_0^1 y_t(i)^\theta di \right)^{\frac{1}{\theta}} \quad (5)$$

where $\theta > 1$. θ determines the elasticity of substitution between the various inputs. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $y_t(i)$, $i \in (0, 1)$ by maximizing the static profit equation

$$\max_{\{X_t(i)\}_{i \in (0,1)}} P_t y_t - \int_0^1 P_t(i) y_t(i) di \quad (6)$$

subject to (5), where $P_t(i)$ denotes the price of intermediate good i . This yields demand functions of the form:

$$y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{\frac{1}{\theta-1}} y_t \quad (7)$$

and the following general price index

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\theta}{\theta-1}} di \right)^{\frac{\theta-1}{\theta}} \quad (8)$$

The final good may be used for consumption — private or public — and investment purposes.

1.3 Intermediate goods producers

Each firm i , $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns-to-scale technology, represented by the production function

$$y_t(i) = a_t k_t(i)^\alpha h_t(i)^{1-\alpha} \text{ with } \alpha \in (0, 1) \quad (9)$$

where $k_t(i)$ and $h_t(i)$ respectively denote the physical capital and the labor input used by firm i in the production process. a_t is an exogenous stationary stochastic technology shock.

As in Rotemberg (1992) and Hairaut and Portier (1993) each intermediate goods producing firms faces a cost of adjusting its nominal price, measured in terms of the finished good

$$\frac{\psi}{2} \left(\frac{P_t(i)}{P_{t-1}(i)} - \bar{\pi} \right)^2 y_t$$

where $\psi \geq 0$ and $\bar{\pi}$ measures the gross steady state inflation rate. The firm then selects its price and production plans in order to maximize profits

$$E_t \left[\sum_{j=0}^{\infty} \Phi_{t+j} \left(\frac{P_t(i)}{P_t} y_t(i) - w_t h_t(i) - z_t k_t(i) - \frac{\psi}{2} \left(\frac{P_t(i)}{P_{t-1}(i)} - \bar{\pi} \right)^2 y_t \right) \right]$$

subject to (7) and (9). $\Phi_{t+j} \propto \beta^j \Lambda_{t+j} P_{t+j}$ is the stochastic discount factor of the firm.

This leads to the following set of optimality conditions

$$\left[\frac{P_t(i)}{P_t} - \xi_t(i) \right] \alpha \frac{y_t(i)}{k_t(i)} = z_t \quad (10)$$

$$\left[\frac{P_t(i)}{P_t} - \xi_t(i) \right] (1 - \alpha) \frac{y_t(i)}{h_t(i)} = w_t \quad (11)$$

$$\begin{aligned} \frac{y_t(i)}{P_t} + \frac{1}{\theta - 1} \frac{\xi_t(i)}{P_t(i)} y_t(i) - \frac{1}{P_{t-1}(i)} \psi \left(\frac{P_t(i)}{P_{t-1}(i)} - \bar{\pi} \right) y_t + \dots \\ \dots + \beta E_t \frac{\Lambda_{t+1} P_{t+1}}{\Lambda_t P_t} \frac{P_{t+1}(i)}{P_t(i)^2} \left(\frac{P_{t+1}(i)}{P_t(i)} - \bar{\pi} \right) y_{t+1} = 0 \end{aligned} \quad (12)$$

where $\xi_t(i)$ is the Lagrange multiplier associated with 7.

1.4 The Government

The government collects taxes from the household, $\tau_t(P_t w_t h_t + P_t z_t k_t)$. These revenues are used to finance government expenditure, g_t . The government budget constraint is

given by

$$\tau_t(P_t w_t h_t + P_t z_t k_t) = P_t g_t \quad (13)$$

g_t follows an exogenous process that will be described in the calibration session. τ_t is a progressive tax rate. In order to take progressivity into account the tax rate is assumed to be an increasing, and concave, function of income, such that

$$\tau_t = \tau(y_t) \text{ with } \tau'(\cdot) > 0, \tau''(\cdot) < 0$$

1.5 The Monetary authorities

We study various policies under precommitment. In particular, we consider the following rules:

1. A standard interest rate rule (à la Henderson–McKibbin–Taylor). Namely,

$$\log R_t = \rho_R \log R_{t-1} + (1 - \rho_R)[\log(\bar{R}) + \kappa_\pi(\log \pi_t - \log \pi_t^*) + \kappa_y(\log y_t - \log y_t^*)] \quad (14)$$

π_t and y_t are actual inflation and output respectively and π_t^* and y_t^* are the inflation and output targets respectively. The output target is set equal to steady state output and the inflation target to the steady state rate of inflation.

2. A rule that perfectly stabilizes the inflation rate at its steady state value.
3. A rule that maximizes the welfare of the representative agent.
4. A rule that minimizes the output gap (actual–efficient output).

The "optimal" rule was computed by postulating a quadratic reaction function of the type

$$\log \left(\frac{M_t}{M_{t-1}} \right) - \log(\gamma^*) = \Gamma X_t + X_t' \Upsilon X_t \quad (15)$$

where $X_t = (\hat{k}_t, \hat{m}_{t-1}, \hat{a}_t, \hat{g}_t, \hat{\zeta}_t)'$ denotes the vector state variables, Γ is a (1×5) vector and Υ is a (5×5) symmetric matrix. The unknown Γ_i and Υ_{ij} , $i, j = 1, \dots, 5$ coefficients were determined by using equation (15) together with the log–quadratic approximation to the solution of the model in a quadratic approximation of the utility function² and then maximizing utility. We chose to formulate the rule in terms of the growth of the money

²In appendix A.3, we provide the interested reader with a detailed explanation of the computation of our welfare approximation. Following Woodford [2003], the approximation is taken around the efficient economy. The use of second order approximation has the advantage of taking into account the effects of variability on the average of the consumption and hours processes.

supply rather than of the interest rate because when we followed the latter approach we often run into numerical difficulties.

It should be stressed that the optimal rule is derived under the assumption that the policymaker formulates this policy "outside" of time and precommits to not reoptimize again. Such rules are called optimal (or optimized) simple precommitment rules. While they may not be time consistent, they are attractive for several reasons. First and most importantly, they are computationally feasible while the fully optimal precommitment rules are often impossible to derive in large, non-linear systems such as ours. A second reason behind the popularity of optimal simple rules is that they provide a compact representation of the policymakers decision rule, yet they tend to perform well relative to more complicated policy rules and relative to the fully optimal precommitment policies (see Dennis [2004]).

1.6 Equilibrium

Definition 1 *A symmetric equilibrium of this economy is a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty} = \{w_t, z_t, P_t, R_t, Q_t\}_{t=0}^{\infty}$ and a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty} = \{\{\mathcal{Q}_t^H\}_{t=0}^{\infty}, \{\mathcal{Q}_t^F\}_{t=0}^{\infty}\}$ with*

$$\begin{aligned}\{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{k_{t+1}, x_t, c_t, D_t, M_t, h_t, \tau_t^g\} \\ \{\mathcal{Q}_t^F\}_{t=0}^{\infty} &= \{y_t, k_t, h_t\}_{t=0}^{\infty}\end{aligned}$$

such that:

- (i) *given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ is a solution to the representative household's problem;*
- (ii) *given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ is a solution to the firms' problem;*
- (v) *given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{P}_t\}_{t=0}^{\infty}$ clears the markets.*
- (vi) *given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, government budget constraint (13) is satisfied and monetary authorities conduct policy according to (14)*

The equilibrium conditions are described in the appendix

2 Calibration

The model is parameterized so that it matches features of the US economy in the post world war II era at a quarterly frequency.

The coefficient of risk aversion in the utility function, σ , is set equal to 1.5. ν is set such that households devote 1/3 of their total time endowment to market activities, implying a value of 0.3777. Households are assumed to discount the future at a 3% annual rate, corresponding to a value for β of 0.9926.

The transactions function $\eta(\cdot)$ is borrowed from Schmidt–Grohe and Uribe [2002].

$$\eta(v_t; \zeta_t) = \zeta_t \left(Av_t + \frac{B}{v_t} - 2\sqrt{AB} \right)$$

ζ_t is a velocity shock whose properties are defined below.

The two parameters, A and B , defining the properties of the transaction cost function, are $A=0.0111$ and $B=0.0752$. This leads to a ratio of consumption purchases to money of 2.88. The velocity shock follows an AR(1) process

$$\log(\zeta_t) = \rho_\zeta \log(\zeta_{t-1}) + (1 - \rho_\zeta) \log(\bar{\zeta}) + \varepsilon_{\zeta,t}$$

with $|\rho_\zeta| < 1$ and $\varepsilon_{\zeta,t} \rightsquigarrow \mathcal{N}(0, \sigma_\zeta^2)$. We use parameter values estimated by Ireland, 2002, namely, $\rho_\zeta = 0.95$ and $\sigma_\zeta = 0.018$. We set $\bar{\zeta}$ equal to unity, which corresponds to the Schmidt–Grohe and Uribe specification.

The nominal growth of the economy is set such that it generates an average quarterly inflation rate over the period of 0.96% per quarter ($\bar{\pi} = 1.0096$). The price adjustment costs parameter, ψ , is set to 10. The quarterly depreciation rate, δ , is 0.025 implying an annual depreciation of about 10%. The value of the capital adjustment cost parameter, φ , is set equal to 1 in our benchmark experiment but we let it vary in our sensitivity analysis from 0.1 to 100.

The values of the parameters of the interest policy rule are borrowed from Taylor (1993), such that $\rho_R = 0$, $k_\pi = 1.5$ and $k_y = 0.5$.

θ is set such that the markup is 10% in the steady state. α , the elasticity of the production function to physical capital, is set such that the model reproduces the US labor share — defined as the ratio of labor compensation to GDP — over the sample period (0.575). This implies a value for α of 0.36.

The technology shock is assumed to follow a stationary AR(1) process of the form

$$\log(a_t) = \rho_a \log(a_{t-1}) + (1 - \rho_a) \log(\bar{a}) + \varepsilon_{a,t}$$

Table 1: Calibration: Benchmark case

Technology		
Capital elasticity of intermediate output	α	0.3611
Capital adjustment costs parameter	φ	1.0000
Depreciation rate	δ	0.0250
Parameter of markup	θ	0.9000
Price Adjustment Costs	ψ	10.0000
Preferences		
Discount factor	β	0.9926
Relative risk aversion	σ	1.5000
CES weight in utility function	ν	0.3777
Parameter of transaction cost (linear)	A	0.0111
Parameter of transaction cost (constant)	B	0.0752
Interest Policy Rule		
Interest rate persistence	ρ_R	0.0000
Reaction to inflation	κ_π	1.5000
Reaction to output	κ_y	0.5000
Taxes		
Average Marginal Income Tax	$E(\tau)$	0.1740
Marginal Income Tax par.	a_0	0.2580
Marginal Income Tax par.	a_1	0.7680
Marginal Income Tax par.	a_2	0.5837
Shocks		
Persistence of technology shock	ρ_a	0.9500
Standard deviation of technology shock	σ_a	0.0079
Persistence of government spending shock	ρ_g	0.9696
Volatility of government spending shock	σ_g	0.0098
Persistence of velocity shock	ρ_ζ	0.9500
Volatility of velocity shock	σ_ζ	0.0180

with $|\rho_a| < 1$ and $\varepsilon_{a,t} \rightsquigarrow \mathcal{N}(0, \sigma_a^2)$. We set $\bar{a} = 1$, $\sigma_a = 0.0079$ and $\rho_a = 0.95$.

The government spending shock³ is assumed to follow an AR(1) process

$$\log(g_t) = \rho_g \log(g_{t-1}) + (1 - \rho_g) \log(\bar{g}) + \varepsilon_{g,t}$$

with $|\rho_g| < 1$ and $\varepsilon_{g,t} \sim \mathcal{N}(0, \sigma_g^2)$. Estimating this process over the sample period leads to a persistence parameter, ρ_g , of 0.9696 and a standard deviation of innovations of $\sigma_g = 0.0098$. The government spending to output ratio is set to its observed sample average, 0.22.

The marginal income tax function is borrowed from Gouveia and Strauss (1994). The advantage of this specification is that it has been estimated on US data and it allows to derive the marginal tax rate. The US tax system is assumed to be well represented by

$$T(y) = a_0(y - (a_2 + y^{-a_1})^{\frac{-1}{a_1}})$$

Therefore, the marginal tax rate is given by

$$\tau(y) = a_0(1 - (1 + a_2 y^{a_1})^{\frac{-(1+a_1)}{a_1}})$$

Gouveia and Strauss (1994) estimated $a_0=0.258$ and $a_1=0.768$. a_2 should be calibrated such as to mimic the marginal tax rate $\tau = 0.174$ for the US.

3 The results

The model is solved using a quadratic log-approximation around the deterministic steady state.

We start by offering some information on the empirical fit of the model. Table 2 reports the standard deviation, $\sigma(\cdot)$, of the main variables as well as correlations with output, $\rho(\cdot, y)$, under progressive, proportional and lump sum taxation for the three monetary policies discussed above: the Henderson-McKibbin-Taylor interest rule (IR), the policy of full (perfect) inflation stabilization at its steady state rate (FPS) and the optimal policy (OMR). The corresponding optimal policies are reported in Table 6. As can be seen, the data-matching ability of the model seems satisfactory and is comparable to that of the other versions of the NNS model used in the literature.

³The logarithm of the government expenditures are first detrended using a linear trend.

Table 2: Second Order Moments

	Data		IR		FPS		OMR	
	$\sigma(\cdot)$	$\rho(\cdot, y)$	$\sigma(\cdot)$	$\rho(\cdot, y)$	$\sigma(\cdot)$	$\rho(\cdot, y)$	$\sigma(\cdot)$	$\rho(\cdot, y)$
Progressive Taxation								
y	1.49	1.00	1.33	1.00	1.25	1.00	1.25	1.00
c	0.80	0.86	1.02	0.84	0.83	0.75	0.80	0.74
x	6.03	0.92	3.79	0.94	3.96	0.94	3.95	0.94
h	0.43	0.69	0.59	0.80	0.51	0.70	0.54	0.67
π	0.16	0.32	1.20	-1.00	0.00	0.00	0.12	0.45
R	0.40	0.21	1.14	-1.00	0.03	0.93	0.10	0.75
M/P	3.10	0.37	6.98	0.99	0.78	0.61	0.18	0.80
Constant Tax Rate								
y	1.49	1.00	1.39	1.00	1.31	1.00	1.30	1.00
c	0.80	0.86	1.04	0.85	0.84	0.75	0.81	0.74
x	6.03	0.92	4.03	0.94	4.18	0.95	4.18	0.95
h	0.43	0.69	0.68	0.85	0.57	0.77	0.61	0.74
π	0.16	0.32	1.26	-1.00	0.00	0.00	0.13	0.45
R	0.40	0.21	1.19	-1.00	0.03	0.94	0.10	0.76
M/P	3.10	0.37	7.27	1.00	0.78	0.60	0.19	0.81

Note: All series are HP-filtered. All data cover the period 1960:1–2002:4, except for hours worked that cover the period 1964:1–2002:2. IR , FPS and OMR correspond to the interest rate rule, the policy of perfect price stabilization policy and the optimal policy.

Table 3 reports the volatility (standard deviation) of inflation, the mark up, and various measures of output and the output gap. Note that unlike table 2, variables are not HP-filtered. Several features stand out.

Table 3: Volatilities

	Progressive Taxation				Constant Tax Rate			
	TR	FYS	FPS	OMR	TR	FYS	FPS	OMR
π	3.71	0.96	0.00	0.22	3.95	1.35	0.00	0.23
μ	1.62	0.70	0.00	1.37	1.67	2.67	0.00	1.37
y	3.94	3.83	3.75	3.77	4.19	3.83	3.98	4.00
y^E	3.83	3.82	3.83	3.83	3.83	3.82	3.83	3.83
y^P	4.08	3.68	3.75	3.65	4.36	3.87	3.98	3.86
$y - y^E$	0.20	0.00	0.19	0.22	0.40	0.00	0.25	0.29
$y - y^P$	0.15	0.18	0.00	0.16	0.17	0.22	0.00	0.18
$y^P - y^E$	0.31	0.20	0.19	0.25	0.56	0.15	0.25	0.19

Note: *IR*, *FYS*, *FPS* and *OMR* correspond to the interest rate rule, the policy of output gap $y_e - y$ stabilization, the policy of perfect price stabilization policy and the optimal policy. μ denotes the markup, y^P is potential-flexible price-output and y^E is the efficient level of output.

First, optimal monetary policy tolerates some deviations of inflation from its steady state rate. These departures, however, tend to be quite small under all tax regimes (in comparison, for instance, to the inflation variability obtained under the interest rate rule). Inflation stabilization, though, does not translate into mark up stabilization (second row in the table). Second, the difference between optimal policy and the policy of perfect price stability is negligible as far as the variability of the various measures of output and the output gap is concerned. The only —modest— exception regards the gap between potential (*i.e.* flexible-price) and actual output. Third, optimal policy tolerates small departures of actual from efficient output. As we will see shortly, forcing actual output to perfectly track the efficient level not only destabilizes actual output and inflation (relative to the optimal policy or the policy of perfect price stabilization) but also leads to a substantial welfare loss. And fourth, the cyclical variability of the inefficiency wedge —the difference between natural and efficient output— is quite modest under all tax regimes and policy rules. Under optimal policy, it increases as we move from proportional to progressive taxation, but it always remains small. This suggests that the presence of an endogenous, significant and cyclically variable distortion such as progressive taxation does not necessarily induce substantial cyclical variation in the inefficiency gap. As Woodford [2003] has conjectured, the closer the comovement of natural and efficient output, the stronger the incentive to force actual output to keep

close to potential output. This is accomplished by stabilizing the inflation rate (or the price level in the absence of steady state inflation and inflation indexation). Nonetheless, to this comovement is less than perfect, it is not optimal to try to eliminate this wedge altogether.

Table 4 reports the correlation of output with inflation, the mark up and the various output gaps. The main point of interest here is that optimal policy removes most *systematic* cyclical variation from the behavior of the mark up but, unlike the perfect inflation stabilization policy, it does not completely eliminate it. Note that mark ups would have been countercyclical under other policies (such as the interest rate rule or money targeting).

Table 4: Correlation with output

	Progressive Taxation				Constant Tax Rate			
	TR	FYS	FPS	OMR	TR	FYS	FPS	OMR
π	-1.00	-0.61	0.00	0.00	-1.00	-0.31	0.00	-0.02
μ	-1.00	0.10	0.00	0.06	-1.00	0.77	0.00	0.06
$y - y^E$	0.56	0.00	-0.36	-0.23	0.92	0.00	0.64	0.61
$y - y^P$	-0.95	0.88	0.00	0.73	-0.95	-0.16	0.00	0.75
$y^P - y^E$	0.85	-0.74	-0.36	-0.68	0.95	0.26	0.64	0.22

Note: *IR*, *FYS*, *FPS* and *OMR* correspond to the interest rate rule, the policy of output gap $y_e - y$ stabilization, the policy of perfect price stabilization policy and the optimal policy. μ denotes the markup, y^P is potential –flexible price– output and y^E is the efficient level of output.

Table 5 reports the welfare comparisons for the three tax regimes and under the four monetary policies considered.

These numbers confirm the comments made earlier in connection to the discussion of table 3. Recall that the optimal policy and the policy of perfect inflation stabilization led to almost identical macroeconomic behavior, with the exception of that of inflation (and the mark up). Consequently, the welfare differences between these two policies turn out to be very small. The advantage of the optimal policy lies in its bringing about a higher average level of consumption. This seems to be due to the fact that the optimal policy leads to a higher –and also smoother– level of real balances and hence to lower transaction costs. Note, also that policies that tolerate large deviations from price stability, such as the interest rate rule, cause large welfare losses. Hence, the popular HMT rule does not approximate optimal policy well. A similar conclusion is reached by Khan King and Wolman, [2003] (KKW henceforth).

Let us now turn to the discussion of the properties of the optimal policy relative to that

Table 5: Welfare Analysis

Case	σ_c	σ_h	σ_{ch}	$E(c)$	$E(h)$	Welfare
Progressive Taxation						
TR	4.57	1.67	-1.76	0.6301	0.3366	-64.484149
FPS	4.00	1.62	-3.23	0.6306	0.3336	-63.990586
FYS	4.15	1.59	-2.91	0.6307	0.3338	-64.017084
OMR	4.00	1.64	-3.21	0.6307	0.3336	-63.988872
Constant Tax rate ($a_1 \rightarrow 0$)						
TR	4.82	1.79	-0.94	0.6307	0.3372	-64.512975
FPS	4.18	1.68	-2.66	0.6307	0.3336	-63.984038
FYS	4.07	1.64	-2.98	0.6303	0.3341	-64.087655
OMR	4.18	1.71	-2.63	0.6308	0.3336	-63.981762

Note: *IR*, *FYS*, *FPS* and *OMR* correspond to the interest rate rule, the policy of output gap $y_e - y$ stabilization, the policy of perfect price stabilization policy and the optimal policy.

of perfect inflation smoothing. A policy of perfect price stabilization involves procyclical monetary response to supply shocks and a countercyclical response to fiscal and velocity shocks. Table 6 shows that the optimal reaction towards supply shocks is less procyclical, towards fiscal shocks less countercyclical and towards velocity shocks more procyclical. As a matter of fact, in the last case, optimal monetary policy is procyclical while it would have been countercyclical under a policy of perfect price stabilization.

Table 6: Policy rules

Progressive Taxation	
FYS	$g_{m,t} = \dots + 0.0595\hat{a}_t + 0.0902\hat{g}_t - 0.0283\hat{\zeta}_t \dots$
FPS	$g_{m,t} = \dots + 0.5512\hat{a}_t - 0.1645\hat{g}_t + 0.0914\hat{\zeta}_t \dots$
OMR	$g_{m,t} = \dots + 0.1611\hat{a}_t - 0.0464\hat{g}_t - 0.1251\hat{\zeta}_t + \dots$
Constant Tax rate ($a_1 \rightarrow 0$)	
FYS	$g_{m,t} = \dots + 0.0695\hat{a}_t + 0.1068\hat{g}_t - 0.0278\hat{\zeta}_t \dots$
FPS	$g_{m,t} = \dots + 0.5413\hat{a}_t - 0.1653\hat{g}_t + 0.0914\hat{\zeta}_t \dots$
OMR	$g_{m,t} = \dots + 0.1610\hat{a}_t - 0.0461\hat{g}_t - 0.1253\hat{\zeta}_t \dots$

Note: *FYS*, *FPS* and *OMR* correspond to the policy of output gap $y_e - y$ stabilization, the policy of perfect price stabilization policy and the optimal policy. \hat{a}_t , \hat{g}_t , $\hat{\zeta}_t$ are the supply, fiscal and velocity shock respectively.

The lower procyclicality means that inflation is allowed to fall following a positive supply shock and that actual output increases by less than in the case of FPS. This pattern is uniform across the three tax regimes (see also the impulse response functions in graphs 1-??). These patterns are different from those obtained by KKW in a more restrictive

version of the NK model than the one used here. In particular, they find that the optimal response to a positive, productivity shock is to let the *price level* decrease persistently but negligibly — that is, inflation should approximately remain constant. And to a positive aggregate demand shock, to let inflation decrease temporarily below its steady state level. In our case, we too get an initial decrease in the inflation rate (relative to trend) following a positive productivity shock, but this is almost immediately reversed with an above trend inflation. Moreover, the change in inflation is more substantial than that in KKW. And for fiscal shocks, we again get a non-monotonic result. The impact effect is higher inflation, but it is quickly followed by a drop to a rate below the steady state rate. Hence, while our results confirm the KKW claim that “[...] *examining the response of optimal policy to productivity and government spending shocks we find that the role of optimal policy, to a first approximation, is to stabilize the price level around trend* [...]”, the actual response to these shocks induces different dynamics in inflation than in KKW.

Before concluding this session we would like to argue that the support for perfect price stability offered by our findings may be somewhat stronger than that available in the existing literature for the following reason. Most of the literature makes use of the Calvo contracting scheme. A well known but unfortunate feature of this scheme is that there exists a non-zero set of firms that almost never adjust their prices. This induces higher price dispersion, which in of no consequence when the model is solved via a first order approximation (see Christiano, Evans and Eichenbaum, 2001). But under a second order approximation, the price dispersion term matters and, when large, it can play a decisive role in the rankings of alternative policies as it heavily penalizes departures from perfect stability. Our approach uses price adjustment costs, so there is no price dispersion term favoring strongly price stability in the second order approximated solution of the model.

4 Conclusions

A policy of strict price stabilization allows the equilibrium in a new Keynesian economy with nominal price rigidities to replicate the flexible price equilibrium. If this equilibrium is approximately efficient, then a policy of strict price stabilization is optimal. In reality, though, the flexible price equilibrium may be significantly inefficient due to the presence of various distortions. And moreover, the degree of inefficiency may be large and also cyclically variable. In this case, there is no longer a presumption that monetary policies

that prevent large and sustained departures from perfect price stability will fare well⁴.

One may defend the presumption for strict price stability by questioning the cyclical relevance of various important economic distortions. For instance, Woodford [2003] has argued that “[...] *while one can think of possible sources of variation in the flexible-price equilibrium level of output that would clearly not be efficient, such as variations in tax distortions or in market power, such factors have not been clearly established as important sources of short run fluctuations in economic activity. Thus while it is certainly possible that substantial disturbances of this kind occur, the matter is far from established [...]*” (p. 454).

Alternatively, one may explicitly incorporate such distortions in the general version of the NNS model and investigate their implications for price stability. This is what we do in this paper. We introduce variable tax distortions (proportional or progressive) as a means of having a large, endogenous, cyclically variable inefficiency in the natural level of output. Our results support the presumption that a policy of perfect inflation smoothing (price stability) is approximately optimal even in this case. And that this obtains even in the absence of price dispersion, a factor that tends to strongly favor policies of price stability.

Nevertheless, we have also argued that the presence of an endogenous, significant and cyclically variable distortion such as taxation does not necessarily induce –and may actually reduce– substantial cyclical variation in the inefficiency gap. And that this consideration may have contributed to our finding that a policy of perfect price stability is approximately optimal even in the presence of all these frictions. It thus remains an open question whether there exist, other endogenous, economically meaningful, large distortions that could induce larger and cyclically more “appropriate” variations in the inefficiency gap (the difference between natural and efficient output) and whether such gaps would justify large, optimal departures from perfect price stability. In Collard and Dellas (2004) we make an attempt to address this question by introducing credit market imperfections (following Bernanke, Gertler and Gilchrist (1999), and Christiano, Motton and Rostagno, (2003)). Our preliminary and tentative findings indicate that while this friction can generate a more cyclically variable wedge it does not seem to severely undermine the case for price stability.

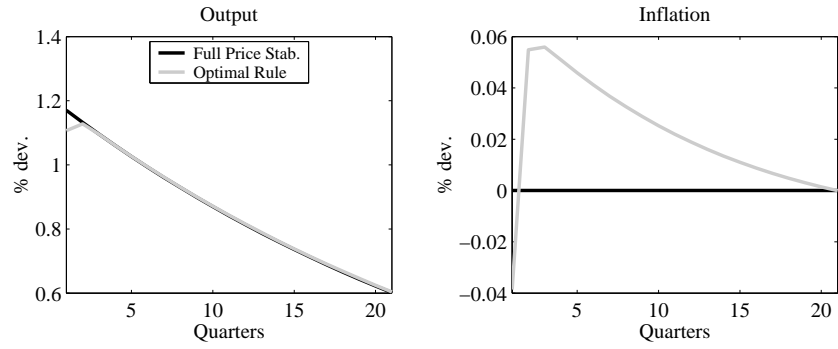
⁴Some results supporting price stability in models with substantial frictions have been obtained but typically within models with very specific, restrictive features and have yet to be confirmed in more general models.

5 References

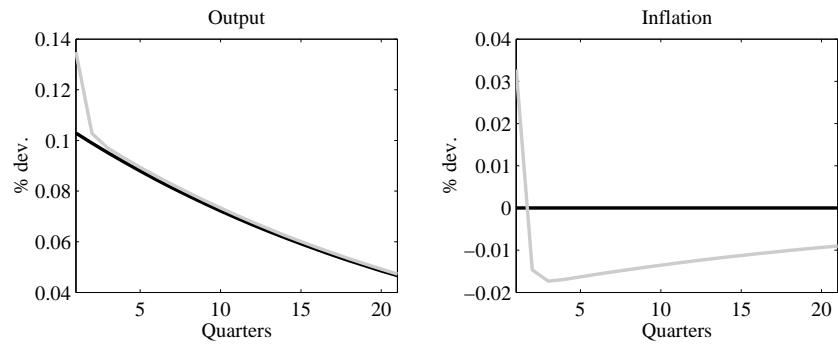
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Figure 1: Impulse Response Functions: Progressive Taxation

(a) Technology Shock



(b) Fiscal Shock



(c) Velocity Shock

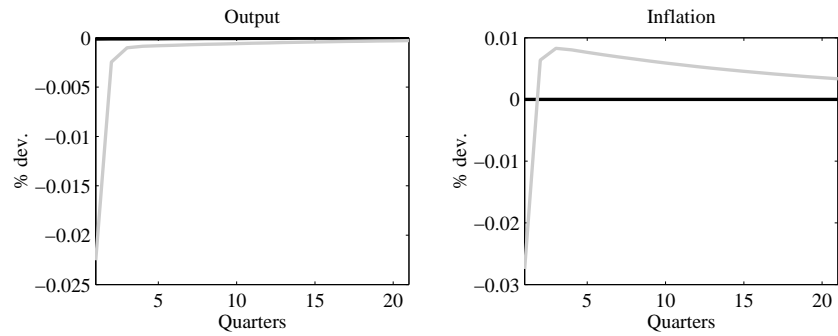
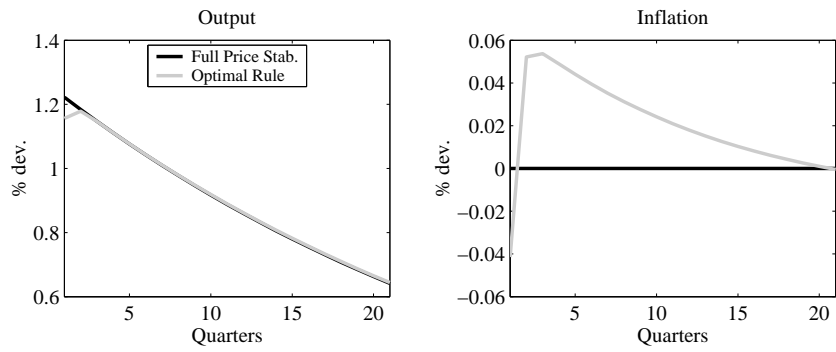
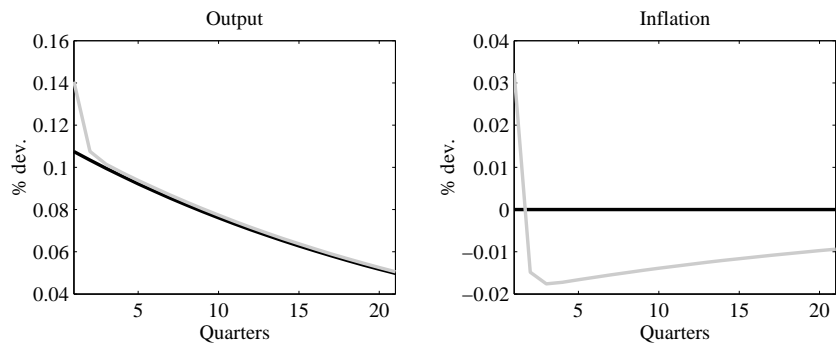


Figure 2: Impulse Response Functions: Constant Tax Rate

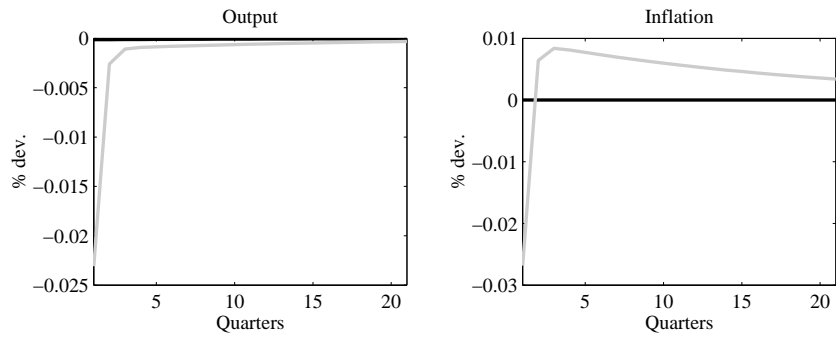
(a) Technology Shock



(b) Fiscal Shock



(c) Velocity Shock



A Appendix

A.1 Description of Equilibrium

$$\begin{aligned}
\nu c_t^{\nu(1-\sigma)-1} (1-h_t)^{(1-\nu)(1-\sigma)} &= \lambda_t (1 + 2\zeta_t (Av_t - \sqrt{AB})) \\
\frac{1-\nu}{\nu} (1 + 2\zeta_t (Av_t - \sqrt{AB})) \frac{c_t}{1-h_t} &= (1-\tau_t)(1-\xi_t)(1-\alpha) \frac{y_t}{h_t} \\
1 - \zeta_t (Av_t^2 - B) &= \frac{1}{R_t} \\
q_t \left(1 - \varphi \left(\frac{x_t}{k_t} - \delta \right) \right) &= \lambda_t \\
y_t &= a_t k_t^\alpha h_t^{1-\alpha} \\
y_t &= (1 + \eta_t) c_t + x_t + g_t + \frac{\psi}{2} (\pi_t - \bar{\pi})^2 y_t \\
k_{t+1} &= x_t - \frac{\varphi}{2} \left(\frac{x_t}{k_t} - \delta \right)^2 k_t + (1 - \delta) k_t \\
\lambda_t &= \beta R_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}} \\
q_t &= \beta E_t \left[\lambda_{t+1} (1 - \tau_{t+1}) (1 - \xi_{t+1}) \alpha \frac{y_{t+1}}{k_{t+1}} + q_{t+1} \left(1 - \delta + \frac{\varphi}{2} \left(\left(\frac{x_{t+1}}{k_{t+1}} \right)^2 - \delta^2 \right) \right) \right] \\
y_t \left(1 + \frac{\xi_t}{\theta - 1} \right) - \pi_t \psi (\pi_t - \bar{\pi}) y_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \pi_{t+1} \psi (\pi_{t+1} - \bar{\pi}) y_{t+1} &= 0 \\
v_t m_t &= c_t
\end{aligned}$$

A.2 Steady state

The steady state of the economy is described by the set of equations

$$\nu \bar{c}^{\nu(1-\sigma)-1} (1 - \bar{h})^{(1-\nu)(1-\sigma)} = \bar{\lambda} (1 + 2\bar{\zeta}(A\bar{v} - \sqrt{A\bar{B}})) \quad (16)$$

$$\frac{1-\nu}{\nu} (1 + 2\bar{\zeta}(A\bar{v} - \sqrt{A\bar{B}})) \frac{\bar{c}}{1-\bar{h}} = (1-\bar{\tau})(1-\bar{\xi})(1-\alpha) \frac{\bar{y}}{\bar{h}} \quad (17)$$

$$1 - \bar{\zeta}(A\bar{v}^2 - B) = \frac{1}{\bar{R}} \quad (18)$$

$$\bar{q} \left(1 - \varphi \left(\frac{\bar{x}}{\bar{k}} - \delta \right) \right) = \bar{\lambda} \quad (19)$$

$$\bar{y} = \bar{a} \bar{k}^\alpha \bar{h}^{1-\alpha} \quad (20)$$

$$\bar{y} = (1 + \bar{\eta}) \bar{c} + \bar{x} + \bar{g} \quad (21)$$

$$\bar{k} = \bar{x} - \frac{\varphi}{2} \left(\frac{\bar{x}}{\bar{k}} - \delta \right)^2 \bar{k} + (1 - \delta) \bar{k} \quad (22)$$

$$\bar{\lambda} = \beta \bar{R} \frac{\bar{\lambda}}{\bar{\pi}} \quad (23)$$

$$\bar{q} = \beta \left[\bar{\lambda} (1 - \bar{\tau})(1 - \bar{\xi}) \alpha \frac{\bar{y}}{\bar{k}} + \bar{q} \left(1 - \delta + \frac{\varphi}{2} \left(\left(\frac{\bar{x}}{\bar{k}} \right)^2 - \delta^2 \right) \right) \right] \quad (24)$$

$$\bar{y} \left(1 + \frac{\bar{\xi}}{\theta - 1} \right) = 0 \quad (25)$$

$$\bar{v} \bar{m} = \bar{c} \quad (26)$$

In what follows, we will assume that \bar{h} , $\bar{\pi}$, $\bar{\tau}$ are known. Equation (25) implies that $\bar{\xi} = 1 - \theta$.

From equation (23), we obtain

$$\bar{R} = \frac{\bar{\pi}}{\beta}$$

From equation (22), we get

$$\bar{x} = \delta \bar{k}$$

which together with (19) implies $\bar{q} = \bar{\lambda}$.

Then, from equation (24), we obtain the capital output ratio

$$\frac{\bar{y}}{\bar{k}} = \frac{1 - \beta(1 - \delta)}{\alpha \beta \theta (1 - \bar{\tau})}$$

we then get

$$\frac{\bar{i}}{\bar{y}} = \delta \frac{\bar{k}}{\bar{y}}$$

From equation (18), we immediately get

$$\bar{v} = \sqrt{\frac{B}{A} + \frac{\bar{R} - 1}{\zeta A \bar{R}}}$$

such that

$$\bar{\eta} = \zeta \left(A\bar{v} + \frac{B}{\bar{v}} - 2\sqrt{AB} \right)$$

The consumption to output ratio is then simply obtained from the good market clearing condition (21)

$$\frac{\bar{c}}{\bar{y}} = \frac{1 - \bar{x}/\bar{y} - \bar{g}/\bar{y}}{1 + \bar{\eta}}$$

Given that \bar{h} is taken to be known, the second equation (17) enables us to obtain parameter ν as

$$\nu = \frac{(1 + 2\bar{\zeta}(A\bar{v} - \sqrt{AB}))\frac{\bar{c}}{\bar{y}}\bar{h}}{(1 + 2\bar{\zeta}(A\bar{v} - \sqrt{AB}))\frac{\bar{c}}{\bar{y}}\bar{h} + (1 - \bar{\tau})(1 - \alpha)\theta(1 - \bar{h})}$$

Finally, equation (20) allows to recover the level of the economy as it yields

$$\bar{y} = \bar{a} \left(\frac{\bar{k}}{\bar{y}} \right)^{\frac{\alpha}{1-\alpha}} \bar{h}$$

Equation (16) may then be used to compute the values of $\bar{\lambda}$.

A.3 Welfare Calculation

Utility can be approximated around the efficient economy by

$$u(c_t, 1 - h_t) \simeq \bar{u} + \bar{u}_c(c_t - \bar{c}) - \bar{u}_\ell(h_t - \bar{h}) + \frac{1}{2}\bar{u}_{cc}(c_t - \bar{c})^2 + \frac{1}{2}\bar{u}_{\ell\ell}(h_t - \bar{h})^2 - \bar{u}_{c\ell}(c_t - \bar{c})(h_t - \bar{h})$$

Further, we can approximate c_t and h_t around the deterministic steady state by

$$c_t \simeq c^* \left(1 + \hat{c}_t + \frac{\hat{c}_t^2}{2} \right) \text{ and } h_t \simeq h^* \left(1 + \hat{h}_t + \frac{\hat{h}_t^2}{2} \right)$$

Therefore

$$c_t - \bar{c} \simeq (c^* - \bar{c}) + c^* \left(\hat{c}_t + \frac{\hat{c}_t^2}{2} \right) \text{ and } h_t - \bar{h} \simeq (h^* - \bar{h}) + h^* \left(\hat{h}_t + \frac{\hat{h}_t^2}{2} \right)$$

We have an approximation of order 2 of the form

$$\begin{aligned} \hat{c}_t &\simeq \pi_c \hat{x}_t + \frac{1}{2} \hat{x}_t' H_c \hat{x}_t + \frac{\varepsilon_c}{2} \\ \hat{h}_t &\simeq \pi_h \hat{x}_t + \frac{1}{2} \hat{x}_t' H_h \hat{x}_t + \frac{\varepsilon_h}{2} \end{aligned}$$

where \hat{x}_t is the vector of state variables. We then have

$$\begin{aligned} E(\hat{c}_t) &= \frac{\eta_c^2}{2} + \frac{\varepsilon_c}{2} \\ E(\hat{c}_t^2) &= \sigma_c^2 + \frac{\varepsilon_c \eta_c^2}{2} + \frac{\varepsilon_c^2}{4} \\ E(\hat{h}_t) &= \frac{\eta_h^2}{2} + \frac{\varepsilon_h}{2} \\ E(\hat{h}_t^2) &= \sigma_h^2 + \frac{\varepsilon_h \eta_h^2}{2} + \frac{\varepsilon_h^2}{4} \\ E(\hat{c}_t \hat{h}_t) &= \sigma_{ch} + \frac{\varepsilon_h \eta_c^2}{4} + \frac{\varepsilon_c \eta_h^2}{4} + \frac{\varepsilon_c \varepsilon_h}{4} \end{aligned}$$

where $\eta_c^2 = H_c E(\hat{x}_t^2)$ and $\eta_h^2 = H_h E(\hat{x}_t^2)$ with $E(\hat{x}_t^2) \equiv \text{vec} E(\hat{x}_t' \hat{x}_t)$. We also have $\sigma_c^2 = \pi_c E(\hat{x}_t \hat{x}_t') \pi_c'$, $\sigma_h^2 = \pi_h E(\hat{x}_t \hat{x}_t') \pi_h'$ and $\sigma_{ch} = \pi_c E(\hat{x}_t \hat{x}_t') \pi_h'$. Note that here we neglected terms of order higher than 2. Then, we have

$$\begin{aligned} E(c_t - \bar{c}) &= (c^* - \bar{c}) + c^* \left(E\hat{c}_t + \frac{E\hat{c}_t^2}{2} \right) \\ E(c_t - \bar{c})^2 &= (c^* - \bar{c})^2 + c^{*2} E\hat{c}_t^2 + 2(c^* - \bar{c})c^* E\hat{c}_t + c^*(c^* - \bar{c}) E\hat{c}_t^2 \\ E(h_t - \bar{h}) &= (h^* - \bar{h}) + h^* \left(E\hat{h}_t + \frac{E\hat{h}_t^2}{2} \right) \\ E(h_t - \bar{h})^2 &= (h^* - \bar{h})^2 + h^{*2} E\hat{h}_t^2 + 2(h^* - \bar{h})h^* E\hat{h}_t + h^*(h^* - \bar{h}) E\hat{h}_t^2 \\ E(c_t - \bar{c})(h_t - \bar{h}) &= (c^* - \bar{c})(h^* - \bar{h}) + h^*(c^* - \bar{c}) E\hat{h}_t + c^*(h^* - \bar{h}) E\hat{c}_t + c^* h^* E\hat{c}_t \hat{h}_t \\ &\quad + h^*(c^* - \bar{c}) \frac{E\hat{h}_t^2}{2} + c^*(h^* - \bar{h}) \frac{E\hat{c}_t^2}{2} \end{aligned}$$

once again neglecting terms of order higher than 2.

These terms all involve computation of expectations which we now detail. The solution of the model takes the form

$$X_{t+1} = M_X X_t + \frac{H_X}{2} X_t^2 + M_E \varepsilon_{t+1} + \frac{\eta_X}{2} \quad (27)$$

$$Y_t = M_Y X_t + \frac{H_Y}{2} X_t^2 + \frac{\eta_Y}{2} \quad (28)$$

where $X_t^2 = \text{vec}(X_t X_t')$. We want to compute first and second order moments for this solution. We first deal with the state equation.

$$E(X_{t+1}) = M_X E(X_t) + \frac{H_X}{2} E(X_t^2) + \frac{\eta_X}{2} \quad (29)$$

implying that

$$E(X_t) = K \left(\frac{\eta_X}{2} + \frac{H_X}{2} E(X_t^2) \right)$$

where $K = (I - M_X)^{-1}$.

$$\begin{aligned}
E(X_{t+1}X'_{t+1}) &= M_X E(X_t X'_t) M'_X + M_X E(X_t X'_t) \frac{H'_X}{2} + M_X E(X_t \varepsilon'_{t+1}) M'_E + M_X E(X_t) \frac{\eta'_X}{2} \\
&\quad + \frac{H_X}{2} E(X_t^2 X'_t) M'_X + \frac{H_X}{2} E(X_t^2 X'_t) \frac{H'_X}{2} + \frac{H_X}{2} E(X_t^2 \varepsilon'_{t+1}) M'_E + \frac{H_X}{2} E(X_t^2) \frac{\eta'_X}{2} \\
&\quad + M_E E(\varepsilon_{t+1} X'_t) M'_X + M_E E(\varepsilon_{t+1} X'_t) \frac{H'_X}{2} + M_E E(\varepsilon_{t+1} \varepsilon'_{t+1}) M'_E + M_E E(\varepsilon_{t+1}) \frac{\eta'_X}{2} \\
&\quad + \frac{\eta_X}{2} E(X'_t) M'_X + \frac{\eta_X}{2} E(X'_t) \frac{H'_X}{2} + \frac{\eta_X}{2} E(\varepsilon'_{t+1}) M'_E + \frac{\eta_X}{2} \frac{\eta'_X}{2} \tag{30}
\end{aligned}$$

Neglecting terms of order greater than 2, acknowledging that $E(\varepsilon_{t+1}) = 0$ and $E(\varepsilon_{t+1} X'_t) = 0$, equation (30) reduces to

$$\begin{aligned}
E(X_{t+1}X'_{t+1}) &= M_X E(X_t X'_t) M'_X + M_X E(X_t) \frac{\eta'_X}{2} + \frac{H_X}{2} E(X_t^2) \frac{\eta'_X}{2} + P \Sigma P' \\
&\quad + \frac{\eta_X}{2} E(X'_t) M'_X + \frac{\eta_X}{2} E(X'_t) \frac{H'_X}{2} + \frac{\eta_X \eta'_X}{4} \tag{31}
\end{aligned}$$

Using the solution for $E(X_t)$, we have

$$\begin{aligned}
E(X_{t+1}X'_{t+1}) &= M_X E(X_t X'_t) M'_X + M_X K \frac{\eta_X \eta'_X}{4} + M_X K \frac{H_X}{2} E(X_t^2) \frac{\eta'_X}{2} + \frac{H_X}{2} E(X_t^2) \frac{\eta'_X}{2} + P \Sigma P' \\
&\quad + \frac{\eta_X}{2} \frac{\eta'_X}{2} K' M'_X + \frac{\eta_X}{2} E(X'_t) \frac{H'_X}{2} K' M'_X + \frac{\eta_X}{2} E(X'_t) \frac{H'_X}{2} + \frac{\eta_X \eta'_X}{4} \tag{32}
\end{aligned}$$

Taking the vector form of the latter equation, we have

$$\text{vec} \Sigma_{XX} = Q^{-1} \text{vec} \left(M_X K \frac{\eta_X \eta'_X}{4} + P \Sigma P' + \frac{\eta_X \eta'_X}{4} K' M'_X + \frac{\eta_X \eta'_X}{4} \right)$$

where $Q \equiv [I_{n_x^2} - M_X \otimes M_X - \frac{1}{4}(\eta_X \otimes M_X K H_X) - \frac{1}{4}(\eta_X \otimes H_X) - \frac{1}{4}(H_X K M_X \otimes \eta_X) - \frac{1}{4}(H_X \otimes \eta_X)]$, and $\Sigma_{XX} \equiv E(X_t X'_t)$. We can then get

$$E(X_t) = K \left(\frac{\eta_X}{2} + \frac{H_X}{2} \text{vec} \Sigma_{XX} \right)$$

From (28), we can compute $E(Y_t)$ and $E(Y_t Y'_t)$ as

$$\begin{aligned}
E(Y_t) &= M_Y E(X_t) + \frac{H_Y}{2} \text{vec} \Sigma_{XX} + \frac{\eta_Y}{2} \\
E(Y_t Y'_t) &= M_Y \Sigma_{XX} M'_Y + M_Y E(X_t) \frac{\eta'_Y}{2} + \frac{H_Y}{2} \text{vec} \Sigma_{XX} \frac{\eta'_Y}{2} + \frac{\eta_Y}{2} E(X'_t) M'_Y + \frac{\eta_Y}{2} (\text{vec} \Sigma_{XX})' \frac{H'_Y}{2} + \frac{\eta_Y \eta'_Y}{4}
\end{aligned}$$