Long and Plosser Meet Bewley and Lucas*

Feng Dong† Yi Wen‡

March, 2018

prepared for the Carnegie Rochester NYU conference on public policy

Abstract

We develop a N-sector business-cycle network model *a la* Long and Plosser (1983), featuring heterogenous money demand *a la* Bewley (1980) and Lucas (1980). Despite incomplete markets and a well-defined distribution of real money balances across heterogeneous households, the enriched N-sector network model remains analytically tractable with closed-form solutions. Relying on the tractability, we establish several important results: (i) The economy's input-output network linkages become endogenously time-varying over the business cycle—thanks to the influence of the endogenous distribution of money demand on cross-sector allocations of commodities. (ii) Despite flexible prices, money is not neutral and monetary injections can generate highly-persistent effects on sectoral output, thanks to the time-varying input-output linkages. (iii) Although money injection is distributed equally across households by design, the real effects are asymmetric across production sectors, e.g., the impact of money is strongest on downstream sectors which purchase intermediate goods from the rest of the economy (such as construction and transportation), but weakest on upstream sectors which supply intermediate goods to the other sectors (such as mining and manufacturing), in sharp contrast to the case of sectoral technology shocks and government spending shocks. Our model also shows that movements in the distribution of money demand could be an important source of the labor wedge documented by the business-cycle accounting literature.

Keywords: Production Networks, Distributional Effect of Monetary Policy, Heterogeneous Money Demand, Incomplete Markets, Time-Varying Velocity of Money, Time-Varying Labor Wedge.

JEL codes: E12, E13, E23, E31, E32, E41, E43, E51.

*Acknowledgment is to be added. The views expressed here are only those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. The other usual disclaim also applies.

†Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai, China. Tel: (+86) 21-52301590. Email: fengdong@sjtu.edu.cn

‡Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166. Office: (314) 444-8559. Fax: (314) 444-8731. Email: yi.wen@stls.frb.org

1 Introduction

The seminal work of Long and Plosser (1983) is among the first to unlock the general-equilibrium properties of the real business cycle (RBC) in a round-about production economy. Their multi-sector RBC model provides a powerful tool to analyze and understand how industry-specific productivity shocks can be propagated through the economy via an input-output network structure. The tractability of the Long-Plosser model also makes the fundamental mechanisms of cross-sectoral work-leisure choices and consumption-saving trade-offs transparent.

However, as emphasized by Long and Plosser (1983), many simplifying assumptions in their model to derive closed-form solutions do come at a cost. For example, since markets are complete, the Long-Plosser model can be easily solved from the perspective of a social planner. But the social-planner approach becomes infeasible when there exist market failures, such as monopolistic competition, sticky prices, externalities, borrowing constraints, and most importantly, heterogenous agents and incomplet markets. Hence, the multi-sector RBC model of Long and Plosser has not achieved the same degree of popularity as the one-sector RBC model of Kydland and Prescott (1982) in the subsequent development of the RBC literature.

In fact, around the same time of the publication of Long and Plosser (1983), many prominent economists were already making efforts to understand the role of money played in the business cycle by developing one-sector rational-expectations models with incomplete markets. Throughout the history of economic thought, money has always been viewed as one of the most important macroeconomic forces to influence aggregate output. Recently, Ramey (2016) shows that monetary policy shocks are still central for our understanding of the business cycle. One of the key challenges in monetary theory was to understand monetary non-neutrality through the lens of time-varying distribution of money demand and fluctuating velocity of money. The cash-in-advance (CIA) models, the famous Baumol-Tobin model and the Bewley model thus became popular. Parallel to Bewley's (1980) seminal contribution, Lucas (1980) was the first to frame the CIA constraint in a heterogenous-agent general-equilibrium framework to study the distribution of money demand without aggregate shocks. But heterogeneity generally renders such models analytically intractable and researchers must rely heavily on numerical methods even in one-sector models. Therefore, it is extremely challenging to analyze how monetary shocks transmit through the economy's input-output network structure under incomplete financial markets.

Our paper tries to bridge this gap by developing a tractable N-sector dynamic-stochasticgeneral-equilibrium (DSGE) model with heterogenous agents and incomplete financial markets. Our model builds on the model of Long and Plosser (1983) by embedding heterogeneous money demand a la Bewley (1980) and Lucas (1980).

We obtain several important results: (i) The distribution of households' money demand and firms' input-output decisions are endogenously related, so that the optimal input-output coefficients for intermediate goods across sectors are time-varying over the business cycle. (ii) Despite flexible prices, money is not neutral and monetary injections can generate highly-persistent effects on sectoral output, thanks to the time-varying input-output linkages. (iii) Although money injection is distributed equally across households by design, the real effects are asymmetric across production sectors, e.g., the impact of money is strongest on the downstream sectors that rely heavily on intermediate goods produced by other sectors (such as construction and transportation), but weakest on the upstream sectors which supply intermediate goods to the rest of the economy (such as mining and manufacturing), in sharp contrast to the case of sectoral technology shocks. Similarly, the fiscal multiplier is the largest by purchasing goods produced by the downstream sectors and weakest by spending on the upstream sectors, suggesting that the multiplier effect is weaker during wars (spending on military equipment) than during peace (spending on infrastructures).

In addition, our model sheds considerable light on the importance of the labor wedge in propagating the business cycle, as shown by Chari, Kehoe, and McGrattan (2007). Using business-cycle accounting based on one-sector representative-agent models, Chari, Kehoe, and McGrattan (2007) show that the measured labor wedge—determined by the gap between the marginal product of labor (MPL) and the marginal rate of substitution (MRS)—accounts for essentially all of the business-cycle fluctuations in the U.S. economy. Moreover, Karabarbounis (2014) finds that, for many countries and especially the United States, fluctuations in the measured labor wedge predominantly reflect movements in the gap between real wage and MRS rather than the gap between real wage and MPL. We complement this literature by showing theoretically that an important source of the measured labor wedge reflects cyclical movements in the distribution of money demand, which create a large gap between real wage and MRS both in the steady state and over the business cycle.[1]

The work most closely related to ours includes Atalay (2017) and Pasten, Schonenle and Weber (2016). Atalay (2017) develops and estimates a multi-industry model with input-output

[1] Bigio and La'O (2016) also obtain an endogenous labor wedge in their multi-sector model but in a static setting and in a real model without money.

linkages.[2] His quantitative analysis indicates that industry-specific shocks can account for at least half of aggregate output volatility. Pasten, Schonenle and Weber (2016) address the propagation of monetary policy shocks in a multi-sector Calvo model with intermediate inputs. Ozdagli and Weber (2017) empirically explore the importance of production networks for the transmission of macroeconomic shocks using stock-market reactions to monetary policy shocks. One of the main differences between this literature and our paper is that we are among the first to study the implications of incomplete financial markets and time-varying distribution of real money balances for the propagation of monetary shocks through input-output network structure without the assumption of sticky prices. We also contribute to this literature by offering an analytically tractable network model with heterogenous agents and incomplete markets. Thus our framework can be readily extended to studying the issues of international trade, optimal capital taxation and optimal government debts (*a la* Aiyagari, 1994, and others) in a multi-sector setting with money and a realistic input-output structure.[3]

Our paper also relates to the growing literature on production networks. In particular, see Dupor (1999), Gabaix (2011), Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012), Acemoglu, Akcigit, Kerr (2016), Acemoglu, Carvalho, and Tahbaz-Salehi (2017) and Oberfield (2017), among others, for the recent progress of network theory. For the application of network theory in macro and finance, see Kim and Shin (2012), Kalemli-Ozcan, Kim, Shin, Sørensen and Yesiltas (2014), Bigio and La'O (2016), Luo (2017) and Su (2017) for recent contributions on the issue of financial shocks to production chains. Also, see Horvath (1998, 2000) and Shea (2002) for their analyses on sectorial shocks and aggregate fluctuations. Also see Baqaee (2017), and Baqaee and Farhi (2017) for analysis of the macroeconomic impact of microeconomic shocks in a production network. However, money is absent from this large literature, and thus there is no room for the discussion of monetary policy. Finally, our tractable heterogenous-agent model with endogenous distribution of money demand is built on the recent works of Wen (2010, 2015) in one-sector models.

2 Revisiting Long and Plosser (1983)

In the original Long-Plosser model, a social planner maximizes expected lifetime utilities by solving

[2] Also see Ando (2014) for measuring US sectoral shocks in the world input–output network.

[3] See Chien and Wen (2017) for a brief literature review on optimal taxation.

$$V\left(S_t\right) = \max \mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s, Z_s\right) \mid \{Y_t, \lambda_t\} \right], \tag{1}$$

subject to the following constant-returns-to-scale Cobb-Douglas production technologies and resource constraints on hours and commodities, respectively:

$$Y_{it} = \lambda_{it} L_{it}^{b_i} \prod_{j=1}^{N} S_{ij,t-1}^{a_{ij}}, \text{ for } i \in \mathbf{N} \equiv \{1, ..., N\}, \tag{2}$$

$$Z_t + \sum_{i=1}^{N} L_{it} = H, \tag{3}$$

$$C_{jt} + \sum_{i=1}^{N} S_{ijt} = Y_{jt}, \ j \in \mathbf{N}, \tag{4}$$

where $b_i + \sum_{j=1}^{N} a_{ij} = 1$, $\{Y_{jt}\}_{j \in \mathbf{N}}$ denotes the $N \times 1$ vector of sectoral output, $\{C_{jt}\}_{j \in \mathbf{N}}$ denotes the $N \times 1$ vector of consumption, S_{ijt} is the quantity of commodity j allocated to producing commodity i in the next period, L_{it} is the hours worked in sector i,[4] Z_t is leisure time, H is total time endowment, and $\{\lambda_{jt}\}_{j \in \mathbf{N}}$ denotes a $N \times 1$ vector of sector-specific productivity shocks.

To recast the social planner's problem into a competitive-market equilibrium, we denote S_{ijt} as the household's savings of intermediate good j to be allocated to sector i as inputs, r_{jit} as the associated real rate of return, w_{jt} as the real wage in sector j, and q_{jt} as the relative price of good j. Then the budget constraint of the representative household is given by

$$\sum_{j=1}^{N} q_{jt} \left(C_{jt} + \sum_{i=1}^{N} S_{ijt} \right) = \sum_{j=1}^{N} q_{jt} \tilde{Y}_{jt}, \tag{5}$$

where $\tilde{Y}_{jt} \equiv \sum_{j=1}^{N} \left(1 + r_{jit}\right) S_{ji,t-1} + w_{jt} L_{jt}$ denotes the household's total market income. To facilitate comparisons with our incomplete-market model in the next section, we assume *quasi-linear* preferences:

$$u\left(C_t, L_t\right) = \sum_{i=1}^{N} \varphi_i \ln C_{it} - \sum_{i=1}^{N} L_{it}, \tag{6}$$

where $\sum_{i=1}^{N} \varphi_i = 1$.

[4] Long and Plosser's (1983) original model assumes a one-period lag in both the intermediate goods and house worked. Here we follow the standard RBC literature by assuming that only intermediate goods enter the production function with a one-period lag (as in the case of capital with a 100% rate of depreciation).

Proposition 1 *The consumption demand for good j, labor supply to sector j, and fraction of good j to be saved as next period's intermediate goods for sector i are given, respectively, by*

$$C_{jt} = \frac{\varphi_j}{\gamma_j} Y_{jt}. \tag{7}$$

$$L_{jt} = \gamma_j b_j, \tag{8}$$

$$S_{ijt} = \beta \frac{\gamma_i a_{ij}}{\gamma_j} Y_{jt}, \tag{9}$$

where the elements γ_j in the vector $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbf{N}}$ are solved by

$$\boldsymbol{\gamma}' = \boldsymbol{\varphi}' (I - \beta \mathbf{A})^{-1}, \tag{10}$$

where $\boldsymbol{\varphi}'$ denotes the $1 \times N$ vector of utility weights $\{\varphi_i\}$, and $\mathbf{A} = (a_{ij})_{N \times N}$ denotes the $N \times N$ matrix of input-output coefficients.

Notice that consumption demand for commodity j is a fixed proportion $\frac{\varphi_j}{\gamma_j}$ of sector j's output, hours worked are constant across sectors, and savings of commodity j to be used as intermediate goods in sector i is also a fixed proportion $\beta \frac{\gamma_i}{\gamma_j} a_{ij}$ of sector j's output.

In particular, the output elasticities of intermediate goods, a_{ij}, enter equation (9); suggesting that the optimal input-output ratio (or the marginal propensity to save intermediate goods i from output produced by sector j), $\frac{S_{ijt}}{Y_{jt}}$, is dictated by a_{ij}, which is the input-output elasticity in the Cobb-Douglas production function (analogous to the golden-rule saving rate). Hence, Long and Plosser (1983) use the US input-output table to calibrate the input-output elasticities $\{a_{ij}\}_{ij \in \mathbf{N}}$ in the production functions.

Finally, using the policy functions to substitute out L_{it} and S_{ijt} in the production function gives the following law of motion for sectoral output:

$$\begin{aligned} \ln Y_{it} &= \ln \lambda_{it} + b_i \ln L_{it} + \sum_{j=1}^{N} a_{ij} \ln S_{ij,t-1} \\ &= \ln \lambda_{it} + \sum_{j=1}^{N} a_{ij} \ln Y_{j,t-1} + b_i \ln (\gamma_i b_i) + \sum_{j=1}^{N} a_{ij} \ln \left(\beta \frac{\gamma_i a_{ij}}{\gamma_j} \right), \end{aligned}$$

which suggests that around the steady state $(\bar{Y})$ the impulse response functions of the output vector $Y_t = (Y_{jt})_{N \times 1}$ have the vector auto-regressive form:

$$\hat{Y}_t = \mathbf{A} \hat{Y}_{t-1} + \hat{\lambda}_t,$$

where $\hat{Y}_t \equiv \log Y_t - \log \bar{Y}$.

3 An N-Sector Bewley-Lucas Model

To extend the Long-Plosser model to a setting with incomplete markets and heterogenous money demand, we introduce heterogenous households with idiosyncratic and uninsurable preference shocks *a la* Lucas (1980). Unlike Lucas (1980), however, we replace the cash-in-advance constraints by the no-short-sale (borrowing) constraints on nominal balances *a la* Bewley (1980, 1983).[5]

There is a continuum of ex ante identical households indexed by $\iota \in [0,1]$. Each household is subject to an idiosyncratic iid preference shock to its marginal utility of consumption, $\theta(\iota)$, which has the distribution $F(\theta) \equiv \Pr[\theta(\iota) \leq \theta]$ with support $[\theta_{\min}, \theta_{\max}]$. Without loss of generality, we normalize the mean of the preference shocks to $\bar{\theta} \equiv \mathbb{E}(\theta) = 1$. Leisure enters the utility function linearly as in Hansen (1985) and Lagos and Wright (2005).[6] Each household chooses a $N \times 1$ consumption vector $\{c(\iota)_j\}_{j \in \mathbf{N}}$, a $N \times 1$ labor supply vector $\{l(\iota)\}_{j \in \mathbf{N}}$, nominal balances $m(\iota)$, and a $N \times N$ matrix of commodity savings $\left\{s'(\iota)_{ij}\right\}_{i,j \in \mathbf{N}}$, to maximize lifetime utility.

To deal with the rate-of-return-dominance problem in portfolio choices, which typically involves monetary assets and interest-bearing assets, and to rule out the possibility of using labor income as a perfect "insurance" device to buffer idiosyncratic preference shocks under quasi-linear preferences, we follow Wen's (2009, 2015) liquidity-demand theory of money by assuming that in each period, household decisions for labor supply and savings of commodities must be made before observing the idiosyncratic preference shock $\theta(\iota)$. Thus, if a household has an urge to consume in period t due to a high realization of $\theta(\iota)$, money stock is the only asset that can be adjusted instantaneously to buffer the random preference shock. This specification implies that money is most liquid compared with other types of savings in meeting consumers' liquidity demand, so money has a liquidity premium and households may find it optimal to hold money as a store of value (in addition to commodity assets) to cope with demand uncertainty, even though money is not essential for exchange and is dominated in rate of return by non-monetary assets (commodity savings). As in the standard literature, however, any aggregate uncertainty is resolved at the beginning of each period and is orthogonal to

[5]Both models feature incomplete heterogeneous agents and financial markets. The only difference between the two models is the specific form of borrowing constraint, in that Bewley imposes the non-negativity constraint $m_t \geq 0$ while Lucas imposes the cash-in-advance constraint $m_t \geq p_t c_t$. As shown by Wen (2010), these two models are equivalent in many of their implications.

[6]The linearity assumption simplifies the model by making the distribution of wealth degenerate. However, unlike Lagos and Wright (2005), the distribution of money holdings in our model is not degenerate but well-defined.

idiosyncratic uncertainty.

3.1 Household Problem

Aggregate shocks are realized in the beginning of each period, after that each household makes decisions on labor supply and commodity savings before observing θ_t. After these decisions are made, the idiosyncratic preference shock θ_t is realized, and each household then chooses consumption and nominal balances.

Specifically, denoting the $N \times 1$ vector of consumption and labor supply of household ι as $\mathbf{c}(\iota)_t = \left\{c(\iota)_{jt}\right\}_{j \in \mathbf{N}}$ and $\mathbf{l}(\iota)_t = \left\{l(\iota)_{jt}\right\}_{j \in \mathbf{N}}$, respectively, real money demand as $\frac{m(\iota)_t}{P_t}$, and the matrix of commodity savings as $\mathbf{S}(\iota)_t = \left\{s(\iota)_{ijt}\right\}_{i,j \in \mathbf{N}}$, the problem of the household is to solve[7]

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(\mathbf{c}(\iota)_t, \mathbf{l}(\iota)_t\right) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{\theta_t(\iota) \cdot \left(\sum_{j=1}^{N} \varphi_j \ln c(\iota)_{jt}\right) - \sum_{j=1}^{N} l(\iota)_{jt}\right\}, \tag{11}$$

subject to the flow of funds constraint

$$\sum_{j=1}^{N} q_{jt}\left(c(\iota)_{jt} + \sum_{i=1}^{N} s(\iota)_{ijt}\right) + \frac{m(\iota)_t}{P_t} = \frac{m(\iota)_{t-1} + \tau_t}{P_t} + \sum_{j=1}^{N} q_{jt}\tilde{y}(\iota)_{jt}, \tag{12}$$

and the no-short-sale (borrowing) constraint on nominal money balances

$$m(\iota)_t \geq 0, \tag{13}$$

where P_t denotes nominal price of aggregate output, τ_t denotes lump-sum aggregate money injection that is equally distributed across households, the utility weight parameters satisfy $\sum_{j=1}^{N} \varphi_j = 1$, with $\varphi_j > 0$ for all $j \in N$, and the household real income from sector $j \in N$ is given by

$$\tilde{y}(\iota)_{jt} \equiv \sum_{i=1}^{N} (1 + r_{jit})\, s(\iota)_{ji,t-1} + w_{jt} l(\iota)_{jt}, \tag{14}$$

which includes both "rental" incomes (returns to savings) and wage incomes. To simplify notations, in what follows we suppress the household index ι unless where confusion may arise.

[7] Similar to Bigio and La'O (2017), we can interpret the basket of consumption goods as a composite condumption good such that $u(c, l) = \theta \cdot \log c - l$, where $c \equiv \prod_{j=1}^{N} c_j^{\varphi_j}$.

3.2 Characterization of Household Decision Rules

To formulate the household problem recursively, we define the household "cash-in-hand" as

$$x_t \equiv \frac{m_{t-1}+\tau_t}{P_t} + \sum_{j=1}^{N} q_{jt}\tilde{y}_{it} - \sum_{j=1}^{N}\sum_{i=1}^{N} q_{jt}s_{ijt}. \tag{15}$$

Notice that all components in x_t (except aggregate state variables) are pre-determined with respect to θ_t in each period (i.e., determined before the realization of θ_t). Then the budget constraint in equation (12) can be rewritten as

$$\sum_{j=1}^{N} q_{jt}c_{jt} + \frac{m_t}{P_t} = x_t. \tag{16}$$

Denoting $J_t(x_t, \theta_t)$ as the value function of the household based on the choices of $\mathbf{c}_t$ and m_{t+1} after the realization of the idiosyncratic preference shock θ_t, we have

$$J_t(x_t,\theta_t) = \max_{\mathbf{c}_t, m_{t+1}} \left\{ \theta_t \cdot \left(\sum_{j=1}^{N} \varphi_j \ln c_{jt} \right) + \beta \mathbb{E}_t V_{t+1}\left(\frac{m_t}{P_{t+1}} \right) \right\}, \tag{17}$$

subject to (16) and (13), where $V_t\left(\frac{m_{t-1}}{P_t}\right)$ is the value function based on the choices of $\mathbf{l}_t$ and $\mathbf{S}_t$ before observing θ_t, which in turn is given by

$$V_t\left(\frac{m_{t-1}}{P_t}\right) = \max_{\mathbf{l}_t, \mathbf{S}_t} \left\{ -\sum_{j=1}^{N} l_{jt} + \int J_t(x_t,\theta_t)\, d\mathbf{F}(\theta_t) \right\}, \tag{18}$$

subject to (15) and $l_{jt} \in \left[0, \bar{l}\right]$ for all $j \in N$.

Proposition 2 *The decision rules follow a cutoff strategy. Denoting θ_t^* as the cutoff for preference shocks and $w_t = q_{jt}w_{jt}$ as the aggregate competitive wage rate (under perfect labor mobility), given prices $\{w_{jt}, r_{ijt}, q_{jt}\}_{i,j\in\mathbf{N}}$, the policy functions of cash-in-hand, consumption, and money demand can be analytically characterized by the following policies:*

$$x_t = w_t\theta_t^* R(\theta_t^*), \tag{19}$$

$$c_{jt} = \frac{\varphi_j}{q_{jt}} \min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} x_t, \text{ for } j \in \mathbf{N}, \tag{20}$$

$$\frac{m_t}{P_t} = \max\left\{ \frac{\theta_t^* - \theta_t}{\theta_t^*}, 0 \right\} x_t, \tag{21}$$

$$\sum_{j=1}^{N} q_{jt}w_{jt}l_{jt} = x_t - \frac{m_{t-1}+\tau_t}{P_t} - \sum_{j=1}^{N} q_{jt}\left[\sum_{i=1}^{N} \left((1+r_{jit})\, s_{ji,t-1} - s_{ijt} \right) \right] \tag{22}$$

where the cutoff θ_t^* is independent of the history of household preference shocks and is determined by the Euler equation for money demand:

$$\frac{1}{w_t} = \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \frac{1}{w_{t+1}} R\left(\theta_t^*\right), \tag{23}$$

in which the liquidity premium of money $R\left(\theta_t^*\right)$ satisfies

$$R\left(\theta_t^*\right) = \int_{\theta_{\min}}^{\theta_{\max}} \max\left\{1, \frac{\theta_t}{\theta_t^*}\right\} d\mathbf{F} \geq \mathbf{1}. \tag{24}$$

Proof: See Appendix. ■

Notice that the cutoff θ_t^* is a sufficient statistic to fully characterize the distribution of household money demand and consumption. Specifically, household real money demand $\frac{m_t}{P_t}$ is zero if the urge to consume is temporarily high when $\theta_t \geq \theta_t^*$, and is a strictly positive fraction of cash-in-hand if the urge to consume is temporarily low when $\theta_t < \theta_t^*$, suggesting that money serves as a precautionary store of value—it provides a self-insurance in case the future urge to consume may be high. The probability of "cash-stockout" depends on the cutoff, θ_t^*, which is endogenously determined by the households.

Equation (23) shows clearly that the cutoff θ_t^* does not depend on the history of individual households' preference shocks, $\{\theta_0, \theta_1, ..., \theta_t\}$, but depends only on the aggregate state of the economy. Consequently, the optimal level of cash-in-hand x_t is also independent of the history of household preference shocks, as revealed in equation (19).

The intuition is that each household can set labor income (in advance of the realization of θ_t) to targeting an optimal level of cash-in-hand, so that x_t is ex anti optimal with respect to the distribution of θ_t. This in turn implies that regardless of the initial value of real money balances $\frac{m_{t-1}}{P_t}$, the household always adjusts labor income to ensure that the cash-in-hand is sufficient (optimal) to meet expected random consumption and money demand. Given that the shock θ_t is iid and the marginal cost of leisure is constant, all households opt to choose the same level of x_t regardless of their initial money balances. On the one hand, too high a level of cash-in-hand implies excessively low probability of a binding liquidity constraint, which is too costly given the positive inflation rate. On the other hand, too low a level of cash-in-hand implies excessively high probability of a binding liquidity constraint, which is also too costly given the large forgone consumption when θ_t may be high. Hence, the optimal level of cash-in-hand x_t is chosen by adjusting labor supply according to the distribution of θ_t so that x_t is the same across households from day one, making it independent of the history of θ_t. This optimal choice of cash-in-hand simultaneously determines the optimal cutoff θ_t^*.

Notice that the first-order condition for labor choices yields $w_t = \int \eta_{it} dF(\theta_t)$, where η_{it} is the Lagrangian multiplier for the household budget constraint in equation (12). Also, the first-order condition with respect to household consumption is $\theta_t u'(c_t) = \eta_t$. Thus, the aggregate real wage equals the average marginal utilities of consumption across households. Denoting $\Lambda_t = \int \eta_{it} dF(\theta)$ as the average marginal utilities of household consumption, then equation (23) can be rewritten in a more convectional form as

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{P_t}{P_{t+1}} R(\theta_t^*), \tag{25}$$

which indicates the intertemporal trade-offs of money holdings, where $\frac{\Lambda_{t+1}}{\Lambda_t}$ pertains to the aggregate marginal-utility ratio or the pricing kernel, and $\frac{P_t}{P_{t+1}}$ is the inverse of the inflation rate. Hence, the expected rate of return to money is given by the discounted inflation-adjusted liquidity premium, $R(\theta^*) > 1$ (for $\theta < \theta_{\max}$). According to equation (24), the liquidity premium decreases with the cutoff θ_t^* because a higher cutoff implies a lower probability of a binding cash constraint; hence, the shadow rate of return to money is lower. In other words, the higher the probability of a binding liquidity constraint, the higher is the liquidity premium of money—because money's value derives purely from its ability to buffer consumption demand shocks despite the fact that its real rate of return (under positive inflation) is dominated strictly by the rate of time preference $1/\beta$.

Assuming a sufficiently large time endowment $\bar{l}$ to ensure interior solution for labor supply across sectors, then it must be true that real wages are equalized across sectors: $q_{jt} w_{jt} = q_{it} w_{it} = w_t$ for all $i, j \in N$. Combining equations (22) and the above no-arbitrage condition on wage gives the household total labor supply as

$$\sum_{j=1}^{N} l_{jt} = \frac{1}{w_t} \left\{ x_t - \frac{m_{t-1} + \tau_t}{P_t} - \sum_{j=1}^{N} q_{jt} \left[\sum_{i=1}^{N} \left((1 + r_{ji,t}) s_{ji,t-1} - s_{ijt} \right) \right] \right\}. \tag{26}$$

3.3 Firms' Problem

As in the Long-Plosser model, the production technology of each commodity i is given by

$$Y_{it} = \lambda_{it} L_{it}^{b_i} \prod_{j=1}^{N} S_{ij,t-1}^{a_{ij}}, \text{ for } i \in \mathbf{N}, \tag{27}$$

with $b_i + \sum_{j=1}^{N} a_{ij} = 1$, where Y_{it} is output of sector i, $S_{ij,t-1}$ is the total fraction of commodity j (savings from all households) allocated to producing commodity i, L_{it} is the total working hours in sector i, and λ_{it} the sectoral productivity shock.

Let $\delta \in [0,1]$ denote the common rate of depreciation for all intermediate goods. Sector i's profit maximization problem is given by

$$\max_{L_{it},S_{ij,t-1}} q_{it}\left(Y_{it}-\sum_{j=1}^{N}\left(r_{ijt}+\delta\right)S_{ij,t-1}-w_{it}L_{it}\right),$$

subject to (27). The FOCs for $(L_{it}, S_{ij,t-1})$ are identical to those in Long and Plosser (1983) and given, respectively, by

$$w_{it} = b_i\frac{Y_{it}}{L_{it}}, \tag{28}$$

$$r_{ijt}+\delta = a_{ij}\frac{Y_{it}}{S_{ij,t-1}}. \tag{29}$$

3.4 Equilibrium Analysis

Aggregation across Households. We use upper-case bold letters to denote aggregate variables across households and lower-case bold letters to denote price vectors. Then given the sequences of vectors $\{\mathbf{q}_t,\mathbf{w}_t,\mathbf{r}_t\}\equiv\{q_{it},w_{it},r_{ijt}\}_{i,j\in\mathbf{N}}$ and the initial distribution of m_0, by integrating individual policy functions in Proposition 2 under the law of large numbers, we can obtain the dynamic system of equations that govern the path of $\{\mathbf{q}_t,\mathbf{r}_t,\mathbf{w}_t,w_t,\mathbf{L}_t,\mathbf{C}_t,\mathbf{S}_t,\mathbf{Y}_t,X_t,\theta_t^*,M_{t+1},P_t\}$ in a competitive equilibrium.

Proposition 3 *The dynamic system of equations to solve for* $\{\mathbf{q}_t,\mathbf{r}_t,\mathbf{w}_t,w_t,\mathbf{L}_t,\mathbf{C}_t,\mathbf{S}_t,\mathbf{Y}_t,X_t,\theta_t^*,M_{t+1},P_t\}$ *are characterized by the following equations:*

$$C_{jt}=\frac{\varphi_j}{q_{jt}}D\left(\theta_t^*\right)X_t,\ \textit{for } j\in\mathbf{N}, \tag{30}$$

$$\frac{M_t}{P_t}=H\left(\theta_t^*\right)X_t, \tag{31}$$

$$X_t=w_t\theta_t^*R\left(\theta_t^*\right), \tag{32}$$

$$\frac{1}{w_t}=\beta\mathbb{E}_t\frac{P_t}{P_{t+1}}\frac{1}{w_{t+1}}R\left(\theta_t^*\right), \tag{33}$$

$$Y_{it}=\lambda_{it}L_{it}^{b_i}\prod_{j=1}^{N}S_{ij,t-1}^{a_{ij}},\ \textit{for } i\in\mathbf{N}, \tag{34}$$

$$\frac{q_{jt}}{w_t}=\beta\mathbb{E}_t\left(1+r_{ij,t+1}\right)\frac{q_{i,t+1}}{w_{t+1}},\ \textit{for } j\in\mathbf{N}, \tag{35}$$

$$L_{jt}=b_j\frac{Y_{jt}}{w_{jt}},\ \textit{for } j\in\mathbf{N}, \tag{36}$$

$$S_{ij,t-1} = \frac{a_{ij}}{r_{ijt}+\delta} Y_{it},\ \textit{for } i,j \in \mathbf{N}, \tag{37}$$

$$w_t = q_{jt} w_{jt},\ \textit{for } j \in \mathbf{N}, \tag{38}$$

$$C_{jt} + \sum_{i=1}^{N} S_{ijt} = Y_{jt} + (1-\delta) \sum_{i=1}^{N} S_{ji,t-1},\ \textit{for } j \in \mathbf{N}, \tag{39}$$

$$X_t = \frac{M_{t-1}+\tau_t}{P_t} + \sum_{j=1}^{N} q_{jt} \left[\sum_{i=1}^{N} \left((1+r_{jit}) S_{ji,t-1} - S_{ijt} \right) \right] + w_t \sum_{j=1}^{N} L_{jt}, \tag{40}$$

$$\bar{M}_t = M_t = M_{t-1} + \tau_t, \tag{41}$$

where $\bar{M}$ denotes aggregate money supply, $D(\theta_t^*) \equiv \int_{\theta_{\min}}^{\theta_{\max}} \min\left(1, \frac{\theta_t}{\theta_t^*}\right) d\mathbf{F}$ reflects the average marginal propensity to consume, and $H(\theta_t^*) \equiv 1 - D(\theta_t^*)$ reflects the average marginal propensity to hold money (the liquidity demand theory of money).

Proof: See Appendix. ■

Consumption Velocity of Money. Define the aggregate consumption across both households and goods sectors as

$$C_t = \sum_{j=1}^{N} q_{jt} C_{it}. \tag{42}$$

Then equation (30) immediately implies

$$C_t = D(\theta_t^*) X_t. \tag{43}$$

Combining equations (31), (43), and the money-market clearing condition (41) yields

$$P_t C_t = M_t \frac{D(\theta_t^*)}{H(\theta_t^*)}.$$

Then the aggregate (consumption) velocity of money is given by

$$v(\theta_t^*) \equiv \frac{P_t C_t}{M_t} = \frac{D(\theta_t^*)}{H(\theta_t^*)} \in (0,\infty). \tag{44}$$

Thus, money velocity in our model is a function only of the *distribution of money demand* across households, which in turn depends on the aggregate state space, including monetary shocks. This time-varying velocity of money with a open support $(0,\infty)$ is in sharp contrast to the representative-agent CIA models which in general imply a constant velocity of one.[8]

[8] Strictly speaking, the lower bound of velocity in equation (44) is given by $\frac{\mathbb{E}(\theta)}{\theta_{\max}-\mathbb{E}(\theta)}$, which goes to zero as $\theta_{\max} \to \infty$.

Remark 1 *Following Jones (2013) and Bigio and La'O (2016), the composite consumption bundle of an individual household can be defined as*

$$c_t(\theta_t, x_t) = \prod_{j=1}^{N} c_{jt}^{\varphi_j} = \frac{\min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} x_t}{q_t}, \tag{45}$$

where the price index q_t *is given by*

$$q_t \equiv \prod_{j=1}^{N} \left(\frac{q_{jt}}{\varphi_j}\right)^{\varphi_j}. \tag{46}$$

Then, as an alternative to equation (42), we can define aggregate consumption as the aggregate valued added final good:

$$C_t \equiv \int c_t(\theta_t, x_t) = \frac{D(\theta_t^*) X_t}{q_t} = D(\theta_t^*) \theta_t^* R(\theta_t^*) \frac{w_t}{q_t}.$$

By normalizing $q_t = 1$*, i.e.,*

$$\sum_{j=1}^{N} \varphi_j \ln q_{jt} = \sum_{j=1}^{N} \varphi_j \ln \varphi_j, \tag{47}$$

and taking into account the fact that $x_t = X_t$*, we immediately obtain that*

$$C_t = D(\theta_t^*) X_t, \tag{48}$$

which coincides with equation (43).

Thus, the aggregate marginal propensity to consuming total cash-in-hand X_t is given by $D(\theta_t^*) \in \left[\frac{\bar{\theta}}{\theta_{\max}}, 1\right]$, and the aggregate marginal propensity to saving the total cash-in-hand in the form of money is $H(\theta_t^*) = 1 - D(\theta_t^*) \in \left[0, \frac{\theta_{\max} - \bar{\theta}}{\theta_{\max}}\right]$. These marginal propensities are time-varying purely because the distribution of households' money demand (θ_t^*) is time-varying. This is in sharp contrast to the complete-market RBC model of Long and Plosser (1983).

Time-Varying Input-Output Coefficients. Our model is even more similar to the Long-Plosser model if we let the rate of depreciation $\delta = 1$. The following proposition shows that we can also express sectoral aggregate consumption C_{jt} as a linear function of sectoral output Y_{jt}, which can then be compared with equation (7) in Long and Plosser (1983).[9]

[9] To better compare with Long and Plosser (1983), we set $\delta = 1$ for the remaining part of the paper. See the Appendix for the details on the more general case in which $\delta \in [0, 1]$.

Proposition 4 *When $\delta = 1$, the aggregate consumption and savings for commodity j are given, respectively, by*

$$C_{jt} = \frac{\varphi_j}{\gamma_{jt}} Y_{jt}, \text{ for } j \in \mathbf{N}, \tag{49}$$

$$S_{ijt} = \frac{\beta \gamma_{it}}{\gamma_{jt}} \mathbf{a}_{ijt} Y_{jt}, \text{ for } i, j \in \mathbf{N}, \tag{50}$$

with

$$\boldsymbol{\gamma}_t' = \boldsymbol{\varphi}' \left(I - \beta \tilde{\mathbf{A}}_t \right)^{-1} \tag{51}$$

where $\boldsymbol{\gamma}_t'$ and $\boldsymbol{\varphi}'$ denote $1 \times N$ vector of $\{\gamma_{it}\}$ and $\{\varphi_i\}$, respectively, and $\tilde{\mathbf{A}}_t = (\mathbf{a}_{ijt})_{N \times N}$ is the adjusted $N \times N$ Input-Output coefficient matrix with $\mathbf{a}_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t (L_{i,t+1}/L_{it})$.

Proof: See Appendix. ■

Comparing equations (7), (9), and (10) in the Long-Plosser model with equations (49), (50) and (51) in our model reveals the shocking similarity of the two models, but also a key difference between them: the optimal input-output ratio $\mathbf{a}_{ijt}$—the saving rate for commodity j to be used as input in sector i—is time-varying in our model but constant in the Long-Plosser model. As a result, the input-output coefficient matrix $\tilde{\mathbf{A}}_t = (\mathbf{a}_{ijt})_{N \times N}$ is time-varying in our model but constant in their model. This in turn implies that the coefficient vector $\boldsymbol{\gamma}_t' = \boldsymbol{\varphi}' \left(I - \beta \tilde{\mathbf{A}}_t \right)^{-1}$ governing the marginal propensity to consumption across all commodities $j \in \mathbf{N}$ is time-varying in our model while constant in the Long-Plosser model.

Labor Dynamics. In our model, the distribution of money demand is characterized by the cutoff θ_t^*. Her we show that it is the time-varying nature of money demand that dictates household labor supply. Recall that $\mathbf{a}_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t (L_{i,t+1}/L_{it})$. The following proposition shows that the dynamics of labor L_{it} $(i \in \mathbf{N})$ is only a function of θ_t^*:

Proposition 5 *Denoting the scaler function $Z(\theta_t^*) \equiv D(\theta_t^*) R(\theta_t^*) \theta_t^*$, the optimal labor demand in our model is given by*

$$\tilde{\mathbf{L}}_t' = \mathbb{E}_t \tilde{\mathbf{L}}_{t+1}' \beta \mathbf{A} + Z(\theta_t^*) \boldsymbol{\varphi}' = \sum_{k=0}^{\infty} (\beta \mathbf{A})^k \mathbb{E}_t Z\left(\theta_{t+k}^*\right) \boldsymbol{\varphi}', \tag{52}$$

where $\mathbf{A} = (a_{ij})_{ij \in N \times N}$ is the standard input-output coefficient matrix, and $\tilde{\mathbf{L}}_t$ is a $N \times 1$ vector of labor with elements $\tilde{L}_{jt} \equiv L_{jt}/b_j$.

Proof: See Appendix. ■

Note two interesting features. First, in our model the optimal saving rate or input-output ratio for intermediate good i produced by sector j, $\mathbf{a}_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t \left(L_{i,t+1}/L_{it} \right)$, as well as the input-output coefficient matrix $\tilde{\mathbf{A}}_t$, reduce exactly to that in the Long-Plosser model when labor becomes constant in the steady state. Hence, in the steady state equations (49), (50) and (51) in our model are identical to equations (7), (9), and (10) in the Long-Plosser model. The only remaining difference is the steady-state levels of labor supply and output, which arises because of incomplete markets with positive steady-state money demand in our model.

Second, equation (52) indicates that the very reason behind a time-varying labor demand in our model is solely because the time-varying distribution of money demand θ_t^*. Labor demand would be constant in our model if the cutoff θ_t^* is constant, either because the variance of the preference shocks degenerates to zero ($\theta_{\min} = \theta_{\max} = \bar{\theta}$), or because households are never borrowing constrained ($\theta_t^* = \theta_{\max}$, such as under the Friedman-rule inflation rate $\pi = \beta - 1$), or because households opt not to hold money at all ($\theta_t^* = \theta_{\min}$, such as in the case of hyper inflation where the liquidity premium reached its maximum at $R\left(\theta_{\min}\right)$. In all such cases the aggregate variables in our monetary model behave exactly like their counterparts in the Long-Plosser model under aggregate shocks, and even their steady-state values are identical (see below for a proof).

Such properties are due to the design of our model—we design our model in such a way so that labor becomes time-varying only because of a time-varying distribution of money demand and nothing else. In other words, there are many ways to make labor time-varying in the Long-Plosser model by introducing new features or frictions, but such modifications can easily destroy the analytical tractability of the Long-Plosser model and loose the closed-form expressions for the input-output coefficients.

Labor demand is time-varying in our model because labor supply is time-varying, which in turn is because the marginal propensities to consume and save are time-varying, which in turn is because household money demand and hence the cutoff θ_t^* are time-varying. Either real or nominal aggregate shocks will inevitably change the household money demand and its distribution because the aggregate price level—which determines the purchasing power and rate of return to money—responds to aggregate shocks. Hence, as long as aggregate price level responds to shocks, the distribution of money balances will be time-varying, and hence the input-output coefficient matrix $\tilde{\mathbf{A}}_t$ will be time-varying.

The reason that the fraction of currently produced commodity j to be saved and allocated to sector i for the next period depends not only on the input-output elasticity a_{ij} in the production technology but also on $\mathbb{E}_t\left(L_{i,t+1}/L_{it}\right)$, the expected increase in sector i's labor input, is as follows. Since labor and intermediate goods are complements, a higher future labor demand in sector i relative to the present implies a higher productivity of all intermediate goods used in sector i. Since it takes one period to reallocate the newly produced intermediate goods across sectors, only the higher expected future increase in labor demand in sector i matters for the productivity of intermediate goods. Hence, given the linear policy function of money demand, the optimal input-output ratio is linear and exactly $\mathbf{a}_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t\left(L_{i,t+1}/L_{it}\right)$; consequently, the input-output coefficient matrix is exactly $\tilde{\mathbf{A}}_t = \left(\mathbf{a}_{ijt}\right)_{ij \in N\times N}$ and the consumption coefficient vector is exactly $\boldsymbol{\gamma}_t' = \boldsymbol{\varphi}'\left(I - \beta\tilde{\mathbf{A}}_t\right)^{-1}$, analogous to their counterparts in the Long-Plosser model.

As already noted, the function $Z\left(\theta_t^*\right) = \left[1 - H\left(\theta_t^*\right)\right] R(\theta_t^*)\theta_t^*$ in the labor equation (52) depends only on the cutoff θ_t^* characterizing the distribution of household money demand. It has three terms to capture the intensive margin and the extensive margin of the aggregate money demand: $H\left(\theta_t^*\right)$ is the marginal propensity to hold money (as in equation (31)), $R\left(\theta^*\right) \geq 1$ is the liquidity premium of money, and θ^* is the cutoff capturing the fraction of cash-constrained households. Hence, $Z\left(\theta_t^*\right)$ pertains to the strength of aggregate money demand as a result of the change in the distribution of money demand. Hence, labor demand (supply) is time varying in our model precisely and only because the distribution of money demand is time varying.

Further, equation (52) suggests that optimal labor demand is itself forward looking: the current demand for labor is a distributed sum of changes in the strength of future distributions of money demand, discounted and compounded by the Long-Plosser input-output coefficient matrix $\mathbf{A}$. This property leads to a large demand-side multiplier effect of aggregate monetary shocks or government-spending shocks on employment, in sharp contrast to cases of sectoral TFP shocks—since the distribution of money demand is not sensitive to TFP shocks, labor is essentially constant, as in the Long-Plosser model (see the dynamic analysis in Section 4 below).

The Labor Wedge. In one-sector representative-agent models without frictions, firm's MPL equals household's MRS. However, this is not true in the data. The literature on business cycle accounting pioneered by Chari, Kehoe, and McGrattan (2007) shows that there is a wedge between the measured MPL (i.e., $\frac{\partial F(K,N)}{\partial N}$) and the measured MRS (i.e., $-u_l/u_c$). They show that this labor wedge accounts for essentially all of the aggregate output fluctuations in the Great Depression and the post-war period when calibrated to a standard one-sector representative-agent real business cycle model. In a recent empirical study, Karabarbounis

(2014) shows that the measured labor wedge comes mainly from the gap between the real wage and the household's MRS.

Motivated by the studies of Chari, Kehoe, and McGrattan (2007) and Karabarbounis (2014), if we define the labor wedge in our model as the difference between the aggregate marginal product of labor ($MPL_t = W_t$) and the aggregate marginal rate of substitution in our model ($MRS_t = -u_{l,t}/u_{c,t} = C_t$ under quasi-linear preference), then the labor wedge through the lens of our model is given by

$$\tau_t^w \equiv \ln W_t - \ln C_t = \ln \frac{1}{D(\theta_t^*)\,\theta_t^* R(\theta_t^*)} \equiv \ln \frac{1}{1 - [1 - Z(\theta_t^*)]} \geq 0, \tag{53}$$

which can be approximated as $\tau_t^w \approx 1 - Z(\theta_t^*) \geq 0$. This wedge vanishes if and only if there is no uninsurable risk (i.e., $Var(\theta) \to 0$), or if the borrowing constraints do not bind (i.e., under the Friedman rule), or if money is not held as a store of value (e.g., under hyper inflation). In each of these cases, we have $Z(\theta_t^*) \to 1$ and thus $\tau_t^w \to 0$ (see more detailed analysis in the next section). This wedge $\tau^w(\theta_t^*)$ is countercyclical, as in the data.

Hence, our model suggests that an important source of the measured labor wedge observed by Chari, Kehoe, and McGrattan (2007) and Karabarbounis (2014) could come from movements in the distribution of household money demand. In the next section we will show that the movements in the model-implied labor wedge in our model is consistent with the empirically measured labor wedge documented by Karabarbounis (2014), which suggests that the measured labor wedge indeed comes mainly from the household side, or from the gap between the observed real wage and measured household MRS (i.e., $-u_l/u_c$).

3.5 Steady-State Analysis

Our model is analytically tractable in the steady state even if the depreciation rate $\delta < 1$.[10] In the steady state, the cutoff value θ^* is determined by

$$R(\theta^*) = \frac{1+\pi}{\beta}, \tag{54}$$

where $\pi \equiv P_{t+1}/P_t - 1$ denotes the steady-state inflation rate. This relationship implicitly solves for the cutoff $\theta^*(\pi)$ as a function of the inflation rate. It shows that the distribution of money demand θ^* depends only on the rate of return to money—captured by the inflation rate. In particular, since $\frac{\partial R}{\partial \theta^*} < 0$, a higher inflation implies a lower cutoff θ^*, and hence a higher probability of a binding liquidity constraint, $\Pr[\theta \geq \theta^*]$. Also, since $\frac{\partial D(\theta^*)}{\partial \theta^*} < 0$, the

[10] See the appendix for the details

velocity of money, $v = \frac{D(\theta^*)}{1-D(\theta^*)}$, is an increasing function of the inflation rate, suggesting that agents reduce and spend money faster under a higher inflation rate so as to mitigate the cost of holding money (inflation tax).

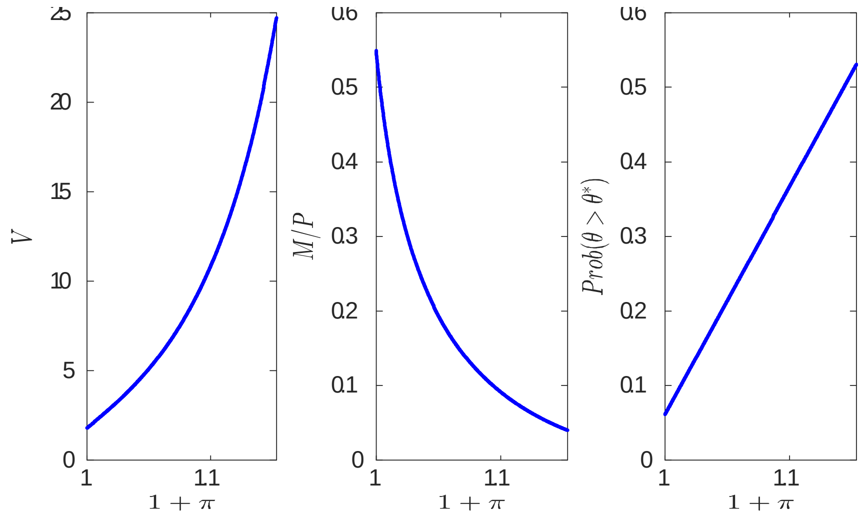

Figure 1. From left to right: the effect of inflation on velocity V, real money demand $\frac{M}{P}$, and the probability of a binding liquidity constraint $\Pr[\theta \geq \theta^*]$.

Under many distributions $F(\theta)$, equation (54) permits a closed-form solution for the cutoff θ^*. For example, if θ follows a Pareto distribution with $F(\theta) = 1 - (\theta/\theta_{\min})^{-\eta}$, $\eta > 1$ and $\mathbb{E}(\theta) = 1$, then equation (54) implies

$$\theta^* = \left[\left(\frac{1+\pi}{\beta} - 1\right)(\eta - 1)\right]^{-\frac{1}{\eta}} \left(1 - \frac{1}{\eta}\right).$$

In such a case, the velocity, money demand, and the portion of population without cash as a function of inflation rate π are graphed in Figure 1.

The wage rate w is given by

$$\log(w) = \frac{\boldsymbol{\varphi}' \left[(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} \mathbf{d} + \ln \boldsymbol{\varphi}\right]}{\boldsymbol{\varphi}' (\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}}, \tag{55}$$

where $\mathbf{d}'$ is a $1 \times N$ vector with elements $d_i = \ln \lambda_i + b_i \ln b_i + \sum_{j=1}^{N} a_{ij} (\ln \beta + \ln a_{ij})$ .Note that both the denominator and the numerator are scalars since $\boldsymbol{\varphi}$, $\mathbf{d}$ and $\mathbf{b}$ are $N \times 1$ vectors. Consequently, $\log(w)$ is a scalar. Given w, the relative prices $(q_i, w_i)_{i \in \mathbf{N}}$ can be solved sequentially by

$$\log \mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1} \cdot (\mathbf{b} \ln (w) - \mathbf{d}), \tag{56}$$

and $w_i = \frac{w}{q_i}$ for $i \in \mathbf{N}$, where $\mathbf{q}' = [\ln q_1, ..., \ln q_N]$.

After solving $\{\theta^*, w, \mathbf{w}, \mathbf{q}\}$, we can obtain the rest of the aggregate (average) variables recursively in the following sequence: $X = w\theta^* R(\theta^*)$, $C_j = \frac{\varphi_j}{q_j} D(\theta^*) X$, $Y_j = \frac{\gamma_j}{\varphi_j} C_j$, $S_{ij} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} Y_i$, for $i, j \in \mathbf{N}$, $L_i = \frac{b_i}{w_i} Y_i$, for $i \in \mathbf{N}$.

The distribution of money demand (or the probability of a binding liquidity constant) creates a labor wedge in our model, as compared with the Long-Plosser model. The labor wedge in turn leads to a output wedge and consumption wedge, supporting the empirical findings that the labor wedge is the most important factor in accounting for aggregate output fluctuations, compared with other wedges such as the investment wedge. The aggregate allocation in our model reduces to that in the Long-Plosser model if and only if the labor wedge vanishes.

As shown earlier, the labor wedge in our model is captured by the factor $Z(\theta^*) = D(\theta^*) R(\theta^*) \theta^* \leq 1$, which is procyclical and appears in each sector j's labor dammed function:

$$L_j = Z(\theta^*) \cdot L_j^{LP}, \tag{57}$$

where $L_j^{LP} = \gamma_j b_j$ denote labor demand in the Long-Plosser model. The output wedge between our model and the Long-Plosser model is captured by

$$\ln Y - \ln Y^{LP} = (\ln Z(\theta^*)) \cdot (I - \mathbf{A})^{-1} \mathbf{b}, \tag{58}$$

where $\ln Y = (\ln Y_1, ..., \ln Y_{\mathbf{N}})'$ and Y^{LP} denotes the counterpart in the Long-Plosser model.

The cutoff $\theta^* \in [\theta_{\min}, \theta_{\max}]$ is interior if and only if household money demand is neither too high nor too low such that the probability of a binding liquidity constraint is strictly between 0 and 1. In other words, households opt to hold money as self-insurance buffer-stock if and only if the inflation rate is neither too high nor too low:

$$\pi_{\min} < \pi < \pi_{\max}, \tag{59}$$

where $\pi_{\min} \equiv \beta - 1$ is the Friedman rule and $\pi_{\max} \equiv \beta \frac{\bar{\theta}}{\theta_{\min}} - 1$ is the maximum inflation rate to induce positive money demand from any household.

The labor-wedge factor $Z(\theta^*) \to 1$, under any one of the three scenarios:

1. the variance of the idiosyncratic shock approaches zero (i.e., $var(\theta) \to 0$ or $\theta_{\min} = \theta_{\max} = \bar{\theta} = 1$),

2. under the Friedman-rule inflation rate ($\pi = \beta - 1$),

3. under hyper inflation $\pi \geq \pi_{\max}$ so that household demand for money is zero.

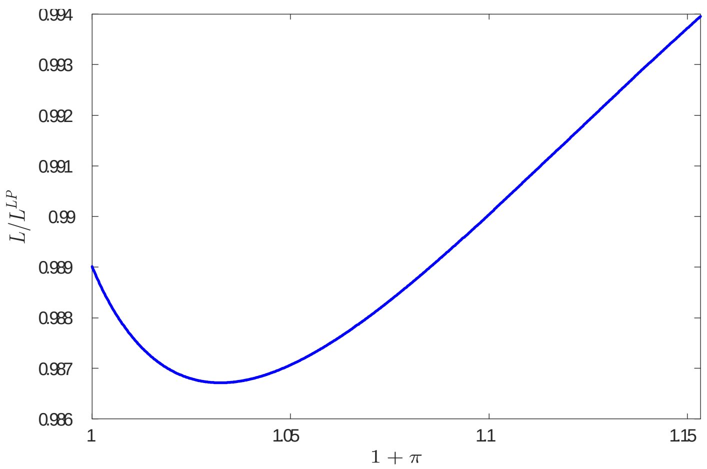

Figure 2a. The effect of inflation on labor-wedge factor $Z(\theta^*) \equiv L_j / L_j^{LP}$.

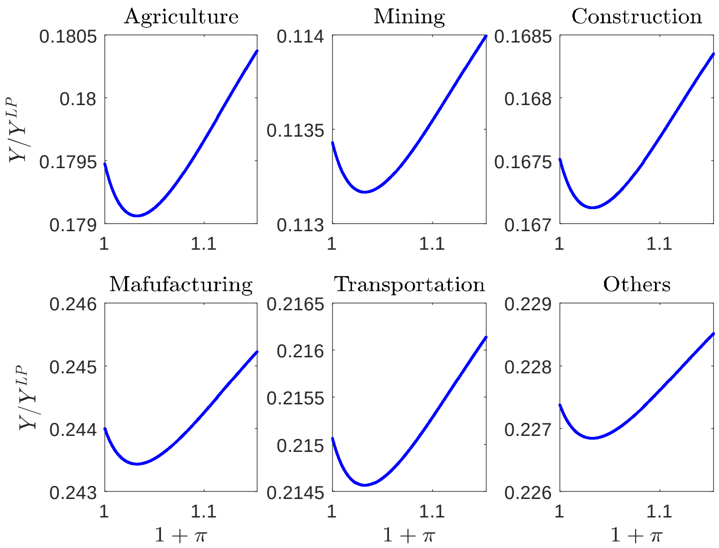

Figure 2b. The distributional effect of inflation on output wedges.

In the first scenario, money is not needed since there is no idiosyncratic uncertainty. At the Friedman rule, the liquidity premium vanishes with $\theta^* = \theta_{\max}$, $R(\theta_{\max}) = 1$, and $\theta_{\max} D(\theta_{\max}) = \bar{\theta} = 1$, so that no household is cash-constrained. Under the hyper inflation rate $\pi \geq \pi_{\max}$, we have $\theta^* = \theta_{\min}$, $R(\theta_{\min}) = \frac{\bar{\theta}}{\theta_{\min}}$, $D(\theta_{\min}) = 1$, and the discounted real rate of return to money $\frac{\beta}{1+\pi_{\max}} R(\theta_{\min}) < 1$, so nobody will hold money (in which case household consumption cannot respond to preference shocks).

Whenever the labor wedge vanishes ($\tau_t^w = 1 - Z(\theta_t^*) \to 0$) under either scenario, the aggregate allocation—the sum of household labor, consumption, savings for intermediate goods, sectoral output—is identical to that in the Long-Plosser model. But the welfare differs across the three scenarios: it is the lowest in the third case because household consumption is not buffered by a store of value when the marginal utility of consumption θ_t changes over time, leading to a lower welfare than the case of no idiosyncratic shocks or no borrowing constraints. This suggests the danger of using representative-agent models to approximate heterogeneous-agent models with incomplete markets, especially when studying optimal government policies (because the social planner or the Ramsey planner must take the distributions into consideration).

However, for an interior solution $\theta^* \in (\theta_{\min}, \theta_{\max})$, we have $Z(\theta^*) < 1$, $L_j < L^{LP}$ and $Y_j < Y_j^{LP}$ for all $j \in \mathbf{N}$. The reason is that holding money imposes a distortionary inflation tax on household income, which reduces household incentives to work. Such an inflation-tax effect vanishes only under the three scenarios discussed above.

The following figures show that as the rate of steady-state inflation increases toward $\pi_{\max}$, beyond which point agents opt not to hold money (because of its low rate of return), then the real allocations in our model converge to those in the Long-Plosser model from below. The same property holds if the rate of inflation decreases toward the Friedman rule. This implies that steady-state ratios of the allocation in our model and that in the Long-Plosser model (e.g., $\frac{L}{L^{LP}}, \frac{Y_i}{Y_i^{LP}}$) are U-shaped with values less than 1 except in the limit $\pi = \pi_{\min}$ or $\pi = \pi_{\max}$.

4 Business-Cycle Analysis

We calibrate the model as follows. As above, we assume θ follows a Pareto distribution with $F(\theta) = 1 - (\theta/\theta_{\min})^{-\eta}$, $\bar{\theta} = \theta_{\min}\eta/(\eta - 1) = 1$, thus $\theta_{\min} = 1 - 1/\eta$. We set the shape parameter $\eta = 2.5$, thus $\theta_{\min} = 0.6$. We also set $\delta = 1$, $\beta = 0.99$, $\pi = 0$, the preference weight $\varphi_i = 1/N$ for $i \in \mathbf{N}$.

Table 1. Parameterization

Parameter	Value	Explanation
β	0.99	time preference
δ	1	depreciation rate
$\varphi_{i\in\mathbf{N}}$	$\frac{1}{6}$	utility weight
π	0	inflation rate
η	2.5	shape parameter
N	6	number of sectors

Table 2. Input-output coefficients $A = (a_{ij})_{6\times 6}$

	Agr.	*Min.*	*Const.*	*Man.*	*Tran.*	*Others*
Agr.	0.4471	0.0033	0.0146	0.2093	0.0999	0.1591
Min.	0.0001	0.0935	0.0427	0.1744	0.0549	0.4854
Cons.	0.0029	0.0104	0.0003	0.4189	0.1209	0.0893
Man.	0.0618	0.0340	0.0050	0.4576	0.0611	0.1267
Tran.	0.0017	0.0004	0.0166	0.1246	0.1040	0.3249
Other	0.0174	0.0212	0.0595	0.1998	0.0871	0.3805

Following Long and Plosser (1983), we use the US input-output (IO) table to calibrate the input-output elasticity parameters a_{ij} in the Cobb-Douglas production function. This is a reasonable approximation in our model since $\tilde{\mathbf{A}}_t = \mathbf{A}$ in the steady state and we evaluate the business-cycle dynamics of our model around the steady state. Table 2 shows that the manufacturing sector uses most heavily all the other sectors' output (including itself) as its inputs. We follow Long and Plosser (1983) by reducing the 15×15 IO table to a smaller IO table with $N = 6$ sectors (i.e., 1. Agriculture, 2. Mining, 3. Construction, 4. Manufacturing, 5. Transportation, 6. Others). The calibrated parameter values are summarized in Table 1, and $A = (a_{ij})_{6\times 6}$, i.e., the condensed 6×6 IO table, is reported in Table 2.

We assume that all aggregate shocks follow AR(1) processes with persistence $\rho = 0.9$. We discuss below the impulse responses of the model to each aggregate shock in turn.

1. Sectorial TFP shocks: $\ln \lambda_{jt} = \rho\lambda_{j,t-1} + \varepsilon_t^{\lambda_j}$.

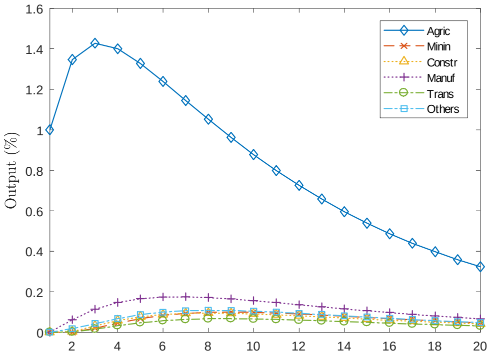

Figure 3a. TFP shock to Agriculture

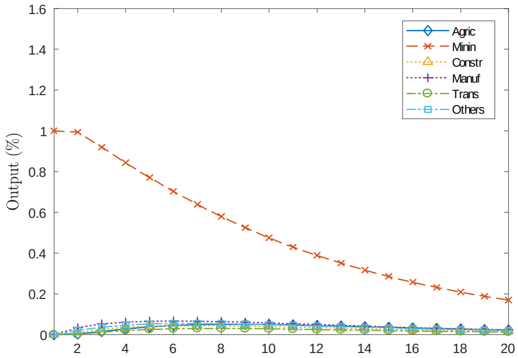

Figure 3b. TFP shock to Mining

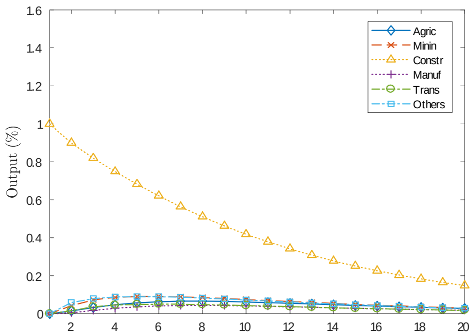

Figure 3c. TFP shock to Construction

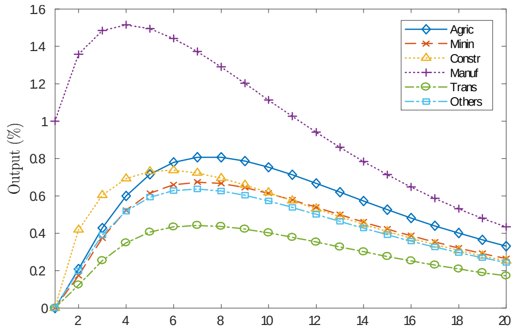

Figure 3d. TFP shock to Manufacturing

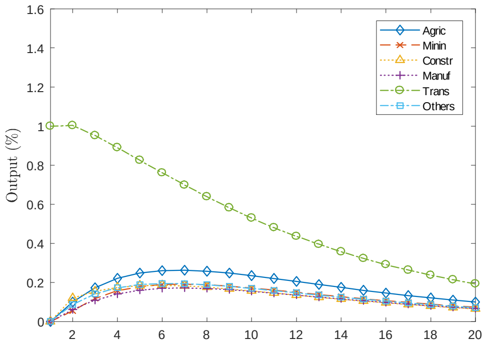

Figure 3e. TFP shock to Transp.

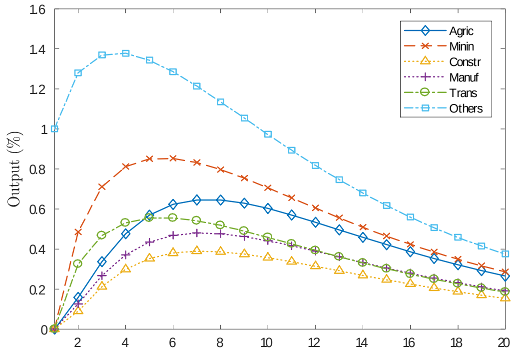

Figure 3f. TFP shock to Others

Recall that labor L_{it} is constant in the Long-Plosser model. In contrast, labor is time-varying in our model due to time-varying distribution of money demand. Therefore there are more amplifications. However, since aggregate price P_t does not respond significantly to TFP shocks in our Bewley-type model, the distribution of money demand also does not move much, making labor essentially constant— the magnitude is in the order of 10^{-3}, so the responses of sectorial output to TFP shocks look very similar to those in the Long-Plosser model, as can be seen below.

The dynamic patterns and the magnitudes of these impulse responses under TFP shocks are nearly identical to those in the Long-Plosser model despite heterogeneous-agent with incomplete markets. To gain intuition, recall that hours worked in our model differs from the Long-Plosser model only by the wedge $Z(\theta_t^*)$ which depends only on the distribution of money demand or the cutoff θ_t^*, so we plot the impulse responses of θ_t^* and $Z(\theta_t^*)$ under sectoral TFP shocks and the monetary shock for comparison purpose. The following figures show that the cutoff θ_t^* and the labor-wedge factor $Z(\theta_t^*)$ are procyclical but change very little (in the order of 10^{-15}) in responding to sectoral TFP shocks. As a result, the distribution of money demand and labor supply remain approximately constant, suggesting that the income effect and substitution effect of TFP shocks on labor supply nearly cancel each other, similar to the Long-Plosser model. This result may lead to the incorrect conclusion that heterogeneity and market incompleteness do not matter for understanding aggregate fluctuations (as argued by Krusell and Smith, 1998).

In sharp contrast, the panels in right-column of Figure 4 (Figure 4a and 4b) show that the cutoff θ^* and the labor-wedge factor $Z(\theta_t^*)$ increase dramatically under a monetary shock (we defer the specification of monetary shocks to the next sub-section), with the order of magnitude 10^{15} times that under TFP shocks (left column in Figure 4a and 4b). Namely, a 10% increase in money supply induces a 6% increase in the cutoff and a 0.16% increase in the labor wedge, compared with a tiny 2.5×10^{-15} percent increase in the cutoff and a similar change in the labor wedge under TFP shocks, suggesting a significantly larger multiplier effect of demand shocks than supply-side shocks.

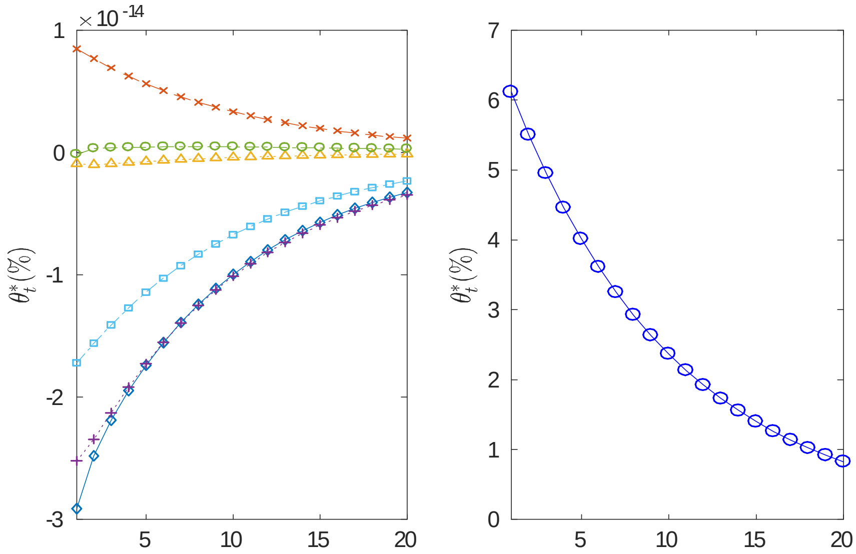

Figure 4a. Impulse response of θ_t^* under sectoral TFP shocks (left) and monetary shock (right), and the legend are referred to that in Figure 4b.

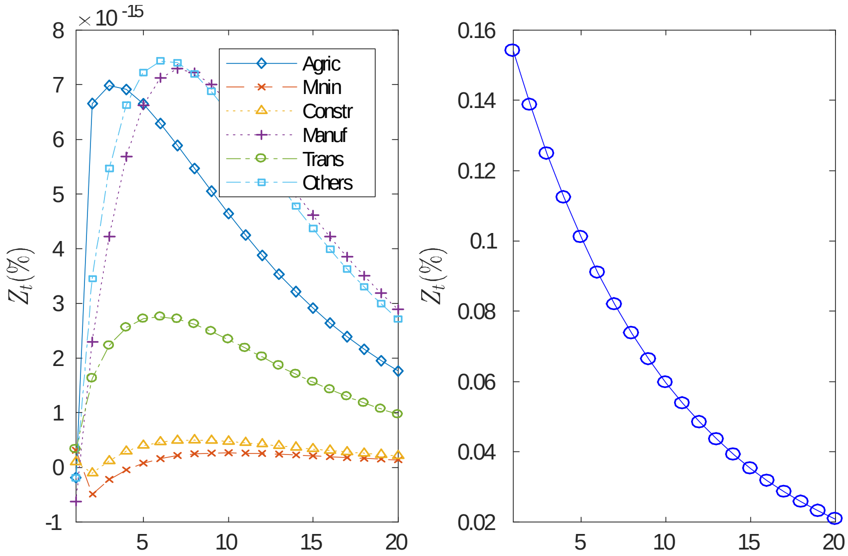

Figure 4b. Impulse response of the labor-wedge factor $Z(\theta_t^*)$ under sectoral TFP shocks (left) and monetary shock (right).

Since the empirically measured labor wedge is the dominating factor explaining the business cycle in the data, and since our model-implied labor wedge $\tau^w(\theta_t^*) \equiv 1 - Z(\theta_t^*)$ is far more volatile and does a better quantitative job in matching the data-implied labor wedge under monetary shocks than under TFP shocks, our model lends support to Ramey (2016)'s observation that monetary policy shocks are central for our understanding of the business cycle.

2. Monetary shock

Money is super-neutral but not neutral in our model. To see this, we assume that aggregate money stock is stationary around the mean $\bar{M}$: $\bar{M}_t = \bar{M} + \bar{\tau}_t$, where money injection $\bar{\tau}_t$ follows an AR(1) process:

$$\bar{\tau}_t = \rho_\tau \bar{\tau}_{t-1} + \varepsilon_t^\tau. \tag{60}$$

Such a specification implies that any injected money is eventually taken out of the economy, as in the US Qualitative Easing episode after the recent financial crisis.[11]

The left panel in Figure 5 shows the responses of aggregate money stock M_t (in red triangles) and aggregate price level P_t (in blue circles). Clearly, aggregate price level does not respond to money supply one-for-one: a 10% increase in the money stock causes only about 2% increase in the price level in the impact period, as if prices are sticky despite flexible prices in our model. The sluggish response in the price level implies that the velocity of money (v_t) declines, as shown in the middle panel. A persistently declining velocity of money also suggests a persistent decrease in the liquidity premium $R\left(\theta_t^*\right)$, which captures the persistent liquidity effect of money observed in the data and helps solve a long-standing puzzle in the monetary theories regarding the liquidity effect of money (see, e.g., Christiano, Eichenbaum, and Evans, 1999, for a literature review on this subject). The right panel shows that the cutoff θ_t^* increases significantly on impact and then declines slowly over time under the monetary injection, suggesting that household real money demand increases sharply and remains high and the probability of a binding liquidity constraint $\Pr\left[\theta \geq \theta_t^*\right]$ drops.

Most importantly, money has significant real effects on sectoral output Y_{it} for all $i \in \mathbf{N}$, as shown in Figure 6a. Notice the strong endogenous multiplier-accelerator effect of money supply shocks in our incomplete-market multi-sector economy: a 10% increase in money supply can generate a significant response in sectoral output across all sectors, with the typical hump-shaped pattern observed in the data. The peak response is reached only 4 quarters after the shock in the manufacturing sector and agricultural sector. The responses of construction and transportation sectors are the strongest while agriculture and manufacturing sectors are the

[11] Since aggregate money demand follows the law of motion, $M_t = M_{t-1} + \tau_t$, then the money market clearing condition, $M_t = \bar{M}_t$, implies that cash received by the households each period is given by $\tau_t = \bar{\tau}_t - \bar{\tau}_{t-1}$, which has an ARMA(1,1) representation: $\tau_t = \rho_\tau \tau_{t-1} + \varepsilon_t^\tau - \varepsilon_{t-1}^\tau$, suggesting that aggregate money demand is also stationary.

weakest, in contrast to the case of sectoral TFP shocks.

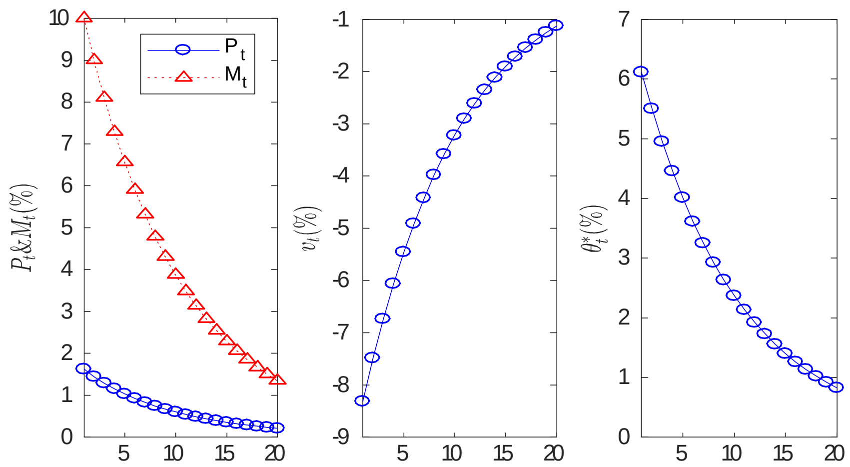

Figure 5. From left to right: impulse responses of the price level (P_t), velocity (v_t), and the cutoff (θ^*_t) to monetary shock.

The reason of such asymmetric affects across sectors are suggested by the input-output table in Table 2. It shows that the manufacturing sector supplies its output to all sectors (including itself) as their inputs with significantly large input-output coefficients (the row entries are relatively large), but does not require too much input from other sectors (the column entries are relatively small); hence TFP shock to this "upstream" sector has a strong "supply-push" effect on the entire economy. On the other hand, the construction (and transportation) sector uses many other sectors' output as its own input (the column entries are relatively large) but is not the main provider of inputs to other sectors (the row entries are relatively small), so this "downstream" sector has a strong "demand-pulling" effect on the entire economy. So monetary shocks act like demand shocks, enticing households to increase savings on commodities produced by the downstream sector(s). Similar rank of sectoral labor responses to monetary shock is revealed in Figure 6b.

Such a monetary non-neutrality originates from the distributional effect of money in the economy: only those households with a binding borrowing constraint will respond to the monetary injection by significantly increasing consumption—because of the relaxation of liquidity shortages, while liquidity-abundant households would hoard the injected money instead of spending it; thus, aggregate price level does not respond one-for-one to the monetary increase, leading to a higher aggregate real demand and output (amplified by sectoral labor demand). The hump-shaped propagation mechanism derives from the input-output linkages amplified by the time-varying nature of the input-output ratios $\mathbf{a}_{ijt}$. As a result, the sectors using other sectors' output more intensively as inputs (i.e., construction and transportation) will respond to the money injection more than the sectors providing output to other sectors as their inputs (i.e., mining and manufacturing). This asymmetric demand-side and supply-side effects happen because of the asymmetric nature of the input-output network (rows vs. columns in the IO Table). The time-varying distribution of money demand also help amplify the asymmetric feature of the IO table through time-varying labor demand.

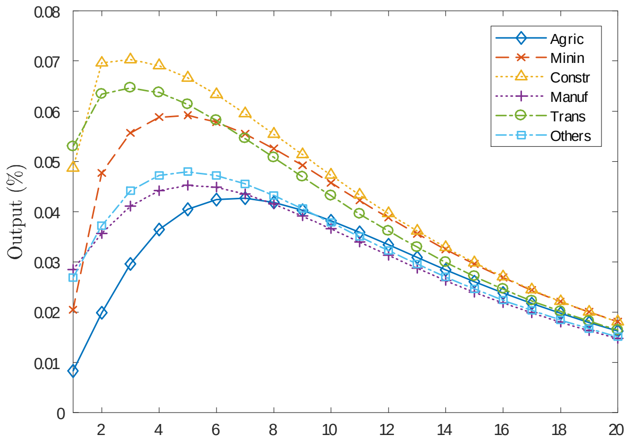

Figure 6a. Impulse response of sectoral output to monetary shock.

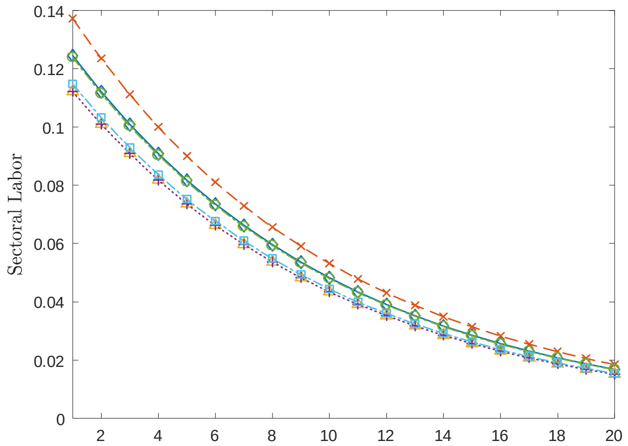

Figure 6b. Impulse responses of sectoral labor to monetary shock.

3. Government spending shock: (to be finished)

This sub-section considers the policy question: If the government can choose the type of goods to purchase, which sector or sectors should it target to maximize the fiscal multiplier? In theory, the government could spread the budget evenly across all sectors, or simply concentrate on one or a few sectors. The answer to this question obviously depends on the nature of the input-output network and is thus the subject of study here.

We introduce government spending in our model in a standard fashion:

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} S_{ijt} + G_{jt}.$$

where for simplicity we set the stead-state ratios $G_j/Y_j = g$ for all j. By re-defining the cash-in-hand x_t to reflect this change in the household budget constraints, we still obtain the following closed-form policy functions:

$$C_{jt} = \frac{\varphi_j}{\gamma_{jt}} Y_{jt}, \text{ for } j \in \mathbf{N},$$

$$S_{ijt} = \frac{\beta \gamma_{it} \mathbf{a}_{ijt}}{\gamma_{jt}} Y_{jt}, \text{ for } i, j \in \mathbf{N},$$

where $\gamma_{jt} \in \boldsymbol{\gamma}_t' = \boldsymbol{\varphi}' \left((1-g) I - \beta \mathbf{A}_t \right)^{-1}$ and $\mathbf{a}_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t \left(L_{i,t+1}/L_{it} \right)$. Let government spending shock follow the log-linear process:

$$\hat{G}_{jt} = \rho_g \hat{G}_{j,t-1} + \varepsilon_t^g \text{ for } i \in \mathbf{N}.$$

Figure 7 shows that the impulse responses of the cutoff to sectoral government spending shocks are identical to each other across sectors, suggesting that sectoral government spending has the same dynamic effects on the distribution of household money demand regardless which sector the spending is targeted.

However, a uniform change in the distribution of household money demand does not imply uniform changes in the sectoral labor and output. Equation (52) suggests that the input-output coefficient matrix also help shape the dynamic responses of labor to aggregate shocks. Figure 8 shows that sectoral government spending shock has the strongest employment effect on the targeted sector.

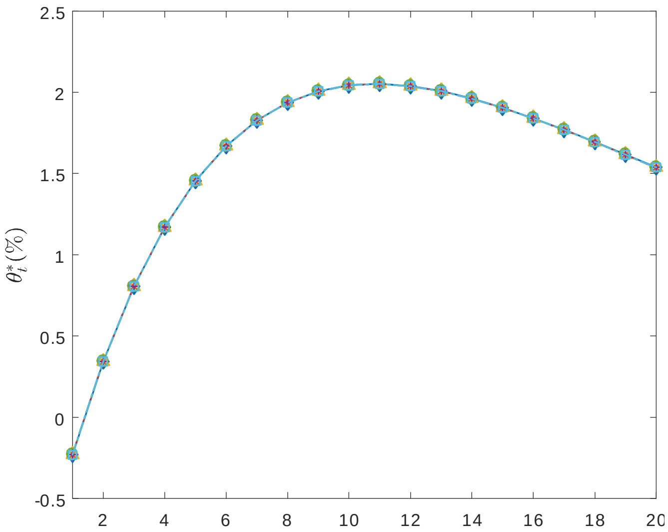

Figure 7. Impulse responses of the cutoff (θ_t^*) to various sectoral government spending shocks.

However, its impact on the other sectors follow the "supply-push" mechanism discussed above, instead of the "demand-pull" mechanism. The intuition is as follows. Although government spending shock is a demand-side shock, but unlike money-supply shock, the higher demand for sector i's output is "taxed" away by the government instead of consumed or saved by the households. As a result, rational households opt to dramatically increase labor supply to the mostly affected sector with government spending so as to minimize its adverse impact on the rest of the economy through the sectoral linkages. In other words, the households treat government spending shock as negative income shock, in contrast to monetary shock. Hence, the upstream sectors such as manufacturing will respond to government spending shocks more strongly than downstream sectors such as mining and construction to mitigate the adverse impact of the shock on the entire economy.

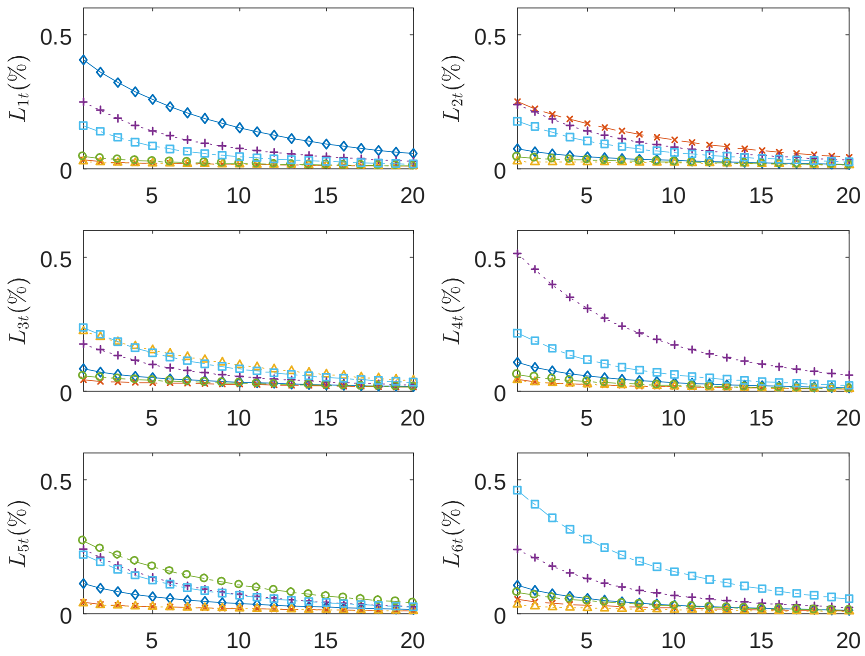

Figure 8. Impulse responses of sectoral labor (from L_{1t} to L_{6t}) to sectoral government spending shocks, where the solid $\Diamond$ denotes the Agriculture sector, long dashed $\times$ denotes the Mining sector, dashed $\triangle$ denotes the Construction sector, dotted $+$ denotes the Manufacturing sector, dashed $\circ$ denotes the Transporation sector, and $\square$ denotes other sectors.

In other words, the demand-side "pulling effect" mechanism does not shed light on the size of the fiscal multiplier on aggregate output. Figure 9 confirms this point by showing that the overall multiplier effect of sectoral government spending on aggregate output is the strongest if government target the upstream manufacturing sector, instead of the downstream construction sector. Figure 9 shows the impulse responses of aggregate output to sectoral government spending shocks. Clearly, the effect of government spending shock on aggregate output is the strongest when applied to the manufacturing sector and the weakest when applied to the mining and transportation sectors. These results suggest that war-time spending on military equipment may have a larger multiplier effect (through manufacturing) than peace-time spending on infrastructure (through construction and transportation).

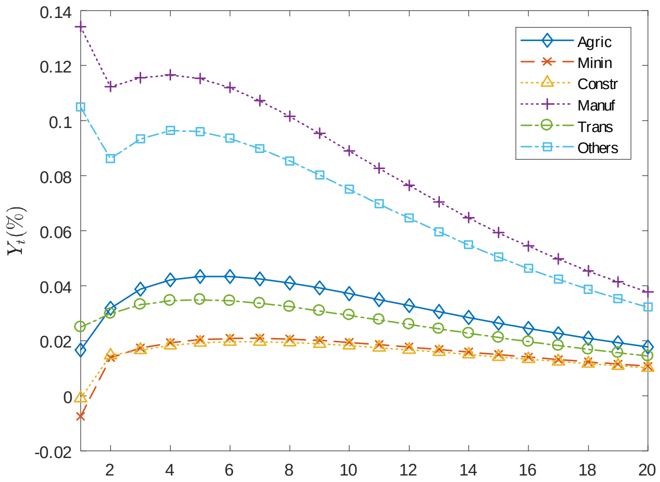

Figure 9. Impulse response of aggregate output (Y_t) to sectoral government spending shocks.

5 Conclusion

In this paper, we extend the seminal N-sector RBC model of Long and Plosser (1983) to a setting with heterogenous money demand and incomplete markets *a la* Bewley (1980) and Lucas (1980). Our enriched model remains analytically tractable as in the original Long-Plosser model. We exploit this tractability to show how the economy's input-output coefficient matrix can be affected by the demand side through a time-varying distribution of household money balances. We then use the model to study a number of issues, including one of the most important issues of monetary theory—the liquidity demand theory of money and its non-neutrality—through the lens of (i) an endogenously time-varying distribution of real money demand and (ii) an endogenously time-varying input-output network. As complementary to the classic Baumol-Tobin model, we show that money is not neutral in the short run and that the time-varying input-output network structure helps propagate monetary shocks through a "demand-pull" channel instead of a "supply-push" channel. In contrast, we find that the impact of government spending on aggregate output depends on the sectors targeted by the government: the fiscal multiplier is larger by targeting the manufacturing sector than by targeting mining, construction and transportation sectors. The reason differs from the conventional Keynesian wisdom of "demand-pulling" effect under insufficient aggregate demand—because the upstream sector (such as manufacturing) provides inputs to all sectors in the economy, hence the private sector's incentive to prevent this sector's intermediate goods supply from falling is the strongest. Hence, both labor supply and money demand will adjust accordingly to accommodate the increase in government spending on manufacturing output, leading to a larger fiscal multiplier. In contrast, monetary shocks have a larger multiplier effect through the downstream sector (such as construction and transportation) than through manufacturing because the downstream sector has a large demand-pulling effect on the economy.

Finally, our model also sheds light on the sources of the measured labor wedge in the business-cycle accounting literature. Chari, Kehoe, and McGrattan (2007) and Karabarbounis (2014) show that the wedge between the observed real wage and measured MRS accounts for essentially all of the aggregate output fluctuations in the U.S. data including the Great Depression. Our model suggests that the measured labor wedge observed by Chari, Kehoe, and McGrattan (2007) and Karabarbounis (2014) could come from movements in the distribution of household money demand under monetary shocks. The fact that monetary shocks are far more important than TFP shocks in triggering movements in both the labor wedge and the distribution of money demand in our model also lends support to Ramey's (2016) observation that monetary policy shocks are central to our understanding of the business cycle. Our future plan is to investigate this issue more closely using empirical data.

Our model can be extended in several directions. First, our work can be readily connected to the literature on inventories by Khan and Thomas (2007) and Wen (2011). Second, our framework can be extended to address international monetary spillover in production networks with data from World Input-Output Tables (WIOD). Third, it could be intriguing to analyze how monetary regimes affect the endogenous formation of the network structure of production (see Acemoglu and Azar, 2017; Oberfield, 2017; Taschereau-Dumouchel, 2017; among others). Finally, our model can be used to study Ramsey optimal taxation problems in production networks. Intuitively, taxing the upstream sector may have a dramatically different welfare effect from that of taxing the downstream sector.

References

Acemoglu, D., Akcigit, U. and Kerr, W., 2016. Networks and the macroeconomy: An empirical exploration. *NBER Macroeconomics Annual*, 30(1), pp.273-335.

Acemoglu, D. and Azar, P.D., 2017. Endogenous Production Networks (No. w24116). National Bureau of Economic Research.

Acemoglu, D., Carvalho, V.M., Ozdaglar, A. and Tahbaz-Salehi, A., 2012. The network origins of aggregate fluctuations. *Econometrica*, 80(5), pp.1977-2016.

Acemoglu, D., Ozdaglar, A. and Tahbaz-Salehi, A., 2017. Microeconomic origins of macroeconomic tail risks. *The American Economic Review*, 107(1), pp.54-108.

Akyol, A., 2004. Optimal monetary policy in an economy with incomplete markets and idiosyncratic risk. *Journal of Monetary Economics*, 51(6), pp.1245-1269.

Alvarez, F., Atkeson, A. and Edmond, C., 2009. Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand. *The Quarterly Journal of Economics*, 124(3), pp.911-967.

Alvarez, F., Atkeson, A. and Kehoe, P.J., 2002. Money, interest rates, and exchange rates with endogenously segmented markets, *Journal of Political Economy*, 110(1), pp.73-112.

Amaral, P.S. and MacGee, J.C., 2017. Monetary shocks and sticky wages in the US great contraction: A multi-sector approach. *Journal of Monetary Economics*, 92, pp.112-129.

Ando, S., 2014. Measuring US sectoral shocks in the world input–output network. *Economics Letters*, 125(2), pp.204-207.

Atalay, E., 2017. How important are sectoral shocks?. *American Economic Journal: Macroeconomics*, 9(4), pp.254-80.

Auclert, A., 2017. Monetary policy and the redistribution channel (No. w23451). National Bureau of Economic Research.

Baqaee, D.R., 2017. Cascading failures in production networks. *Econometrica*, forthcoming.

Baqaee, D.R. and Farhi, E., 2017. The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem (No. w23145). National Bureau of Economic Research.

Barrot, J.N. and Sauvagnat, J., 2016. Input specificity and the propagation of idiosyncratic shocks in production networks. *The Quarterly Journal of Economics*, 131(3), pp.1543-1592.

Bartelme, D. and Gorodnichenko, Y., 2015. Linkages and economic development (No. w21251). National Bureau of Economic Research.

Bewley, T., 1980. The optimum quantity of money. *Models of monetary economies*, ed. by John Kareken and Neil Wallace. Minneapolis, Minnesota: Federal Reserve Bank, pp.169-210

Bewley, T., 1983. A difficulty with the optimum quantity of money. *Econometrica*, 51(5), pp.1485-1504.

Bigio, S. and La'O, J., 2016. Financial frictions in production networks (No. w22212). National Bureau of Economic Research.

Boehm, C., Flaaen, A. and Pandalai-Nayar, N., 2015. Input linkages and the transmission of shocks: Firm-level evidence from the 2011 Tōhoku earthquake. Working Paper, Federal Reserve Board.

Carvalho, V.M., 2014. From micro to macro via production networks. *The Journal of Economic Perspectives*, 28(4), pp.23-47.

Carvalho, V.M., Nirei, M., Saito, Y.U. and Tahbaz-Salehi, A., 2016. Supply chain disruptions: Evidence from the great east japan earthquake. Working Paper, Northwestern University.

Chari, V.V., Kehoe, P.J. and McGrattan, E.R., 2007. Business cycle accounting. *Econometrica*, 75(3), pp.781-836.

Chiu, J. and Molico, M., 2010. Liquidity, redistribution, and the welfare cost of inflation. *Journal of Monetary Economics*, 57(4), pp.428-438.

Christiano, L.J. and Eichenbaum, M., 1995. Liquidity effects, monetary policy, and the business cycle. *Journal of Money, Credit and Banking*, 27(4), pp.1113-1136.

Christiano, L.J., Eichenbaum, M. and Evans, C.L., 1999. Monetary policy shocks: What have we learned and to what end?. *Handbook of Macroeconomics*, 1, pp.65-148.

Ciccone, A., 2002. Input chains and industrialization. *The Review of Economic Studies*, 69(3), pp.565-587.

Conley, T.G. and Dupor, B., 2003. A spatial analysis of sectoral complementarity. *Journal of Political Economy*, 111(2), pp.311-352.

Di Giovanni, J., Levchenko, A.A. and Méjean, I., 2014. Firms, destinations, and aggregate fluctuations. *Econometrica*, 82(4), pp.1303-1340.

Dupor, B., 1999. Aggregation and irrelevance in multi-sector models. *Journal of Monetary Economics*, 43(2), pp.391-409.

Foerster, A.T., Sarte, P.D.G. and Watson, M.W., 2011. Sectoral versus aggregate shocks: A structural factor analysis of industrial production. *Journal of Political Economy*, 119(1), pp.1-38.

Gabaix, X., 2011. The granular origins of aggregate fluctuations. *Econometrica*, 79(3), pp.733-772.

Heathcote, J., Storesletten, K. and Violante, G.L., 2009. Quantitative macroeconomics with heterogeneous households. *Annual Review of Economics*, 1(1), pp.319-354.

Horvath, M., 1998. Cyclicality and sectoral linkages: Aggregate fluctuations from independent sectoral shocks. *Review of Economic Dynamics*, 1(4), pp.781-808.

Horvath, M., 2000. Sectoral shocks and aggregate fluctuations. *Journal of Monetary Economics*, 45(1), pp.69-106.

Huang, K.X. and Liu, Z., 2001. Production chains and general equilibrium aggregate dynamics. *Journal of Monetary Economics*, 48(2), pp.437-462.

Jones, C.I., 2011. Intermediate goods and weak links in the theory of economic development. *American Economic Journal: Macroeconomics*, 3(2), pp.1-28.

Jones, C.I., 2013. Misallocation, input-output economics, and economic growth. In *Advances in economics and econometrics: Tenth world congress* (Vol. 2, pp. 419-58). Cambridge University Press Cambridge, UK.

Kalemli-Ozcan, S., Kim, S.J., Shin, H.S., Sørensen, B.E. and Yesiltas, S., 2014, January. Financial shocks in production chains. In American Economic Association meetings, January.

Kaplan, G., Moll, B. and Violante, G.L., 2016. Monetary policy according to HANK (No. w21897). National Bureau of Economic Research, *The American Economic Review*, forthcoming.

Karabarbounis, L., 2014. The labor wedge: MRS vs. MPN. *Review of Economic Dynamics*, 17(2), pp.206-223.

Kelly, B., Lustig, H. and Van Nieuwerburgh, S., 2013. Firm volatility in granular networks (No. w19466). National Bureau of Economic Research.

Kim, S.J. and Shin, H.S., 2012. Sustaining production chains through financial linkages. *The American Economic Review*, 102(3), pp.402-406.

Kiyotaki, N. and Moore, J., 1997. Credit chains. Working Paper, LSE.

Krueger, D., Mitman, K. and Perri, F., 2016. Macroeconomics and household heterogeneity. In *Handbook of Macroeconomics* (Vol. 2, pp. 843-921). Elsevier.

Kydland, F.E. and Prescott, E.C., 1982. Time to build and aggregate fluctuations. *Econometrica*, 50(6), pp.1345-1370.

Lagos, R. and Wright, R., 2005. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3), pp.463-484.

Levine, D.K., 2012. Production chains. *Review of Economic Dynamics*, 15(3), pp.271-282.

Long Jr, J.B. and Plosser, C.I., 1983. Real business cycles. *Journal of Political Economy*, 91(1), pp.39-69.

Long Jr, J.B. and Plosser, C.I., 1987. Sectoral vs. aggregate shocks in the business cycle. *The American Economic Review*, 77(2), pp.333-336.

Lorenzoni, G., 2011. News and aggregate demand shocks. *Annual Review of Economics*, 3(1), pp.537-557.

Lucas Jr, R.E., 1975. An equilibrium model of the business cycle. *Journal of Political economy*, 83(6), pp.1113-1144.

Lucas Jr„ R.E., 1977. Understanding business cycles. In *Carnegie-Rochester conference series on public policy*, Vol. 5, pp. 7-29. North-Holland.

Lucas Jr„ R.E., 1980. Equilibrium in a pure currency economy. *Economic inquiry*, 18(2), pp.203-220.

Luo, S., 2017. Propagation of financial shocks in an input-output economy with trade and financial linkages of firms. Working Paper, Virginia Tech.

Nakamura, E. and Steinsson, J., 2010. Monetary non-neutrality in a multisector menu cost model. *The Quarterly Journal of Economic*s, 125(3), pp.961-1013.

Oberfield, E., 2017. A theory of input-output architecture. *Econometrica*, forthcoming.

Ozdagli, A. and Weber, M., 2017. Monetary policy through production networks: Evidence from the stock market (No. w23424). National Bureau of Economic Research.

Pasten, E., Schoenle, R. and Weber, M., 2016. The Propagation of Monetary Policy Shocks in a Heterogeneous Production Economy. Working Paper, Chicago Booth.

Pasten, E., Schoenle, R. and Weber, M., 2017. Price rigidities and the granular origins of aggregate fluctuations (No. w23750). National Bureau of Economic Research.

Ramey, V.A., 2016. Macroeconomic shocks and their propagation. *Handbook of Macroeconomics*, 2, pp.71-162.

Taschereau-Dumouchel, M., 2017. Cascades and fluctuations in an economy with an endogenous production network. Working paper, Wharton.

Shea, J., 2002. Complementarities and comovements. *Journal of Money, Credit, and Banking*, 34(2), pp.412-433.

Su, H.L., 2017. Financial frictions, capital misallocation, and input-output linkages. Working paper, NTU.

Wen, Y., 2010. Optimal money demand in a heterogeneous-agent cash-in-advance economy. Working Papers 2010-014, Federal Reserve Bank of St. Louis.

Wen, Y., 2015. Money, liquidity and welfare. *European Economic Review*, 76, pp.1-24.

Williamson, S.D., 2008. Monetary policy and distribution. *Journal of Monetary Economics*, 55(6), pp.1038-1053.

Appendix

A Proofs

Proof of Proposition 1: The Lagrangian is given by

$$\mathcal{L} = u(C_t, L_t) + \mu_t \left[\sum_{j=1}^{N} q_{jt} \tilde{Y}_{jt} - \sum_{j=1}^{N} q_{jt} \left(C_{jt} + \sum_{i=1}^{N} S_{ijt} \right) \right]. \tag{A.1}$$

The first-order conditions (FOCs) with respect to $\{C_{jt}, L_{jt}, S_{jit}\}$ are given, respectively, by

$$\frac{\varphi_j}{C_{jt}} = q_{jt}\mu_t \text{ for } j \in \mathbf{N}, \tag{A.2}$$

$$1 = \mu_t q_{jt} w_{jt} \text{ for } j \in \mathbf{N},$$

$$\mu_t q_{it} = \beta \mu_{t+1} q_{j,t+1} (1 + r_{ji,t+1}) \text{ for } i, j \in \mathbf{N}.$$

$$C_{it} = \frac{\varphi_i}{\gamma_i} Y_{it}$$

Perfect labor mobility across sectors implies a common wage rate:

$$w_t \equiv q_{jt} w_{jt} = \frac{1}{\mu_t} \text{ for } j \in \mathbf{N}. \tag{A.3}$$

The FOCs on L_{jt} and S_{jit} then become

$$\frac{\varphi_j}{C_{jt}} = \frac{1}{w_{jt}}, \tag{A.4}$$

$$\frac{\varphi_i}{C_{it}} = \beta \frac{\varphi_j}{C_{j,t+1}} (1 + r_{ji,t+1}). \tag{A.5}$$

Each sector has a representative firm. Firm j's maximization problem at $t+1$ is given by

$$\max_{X_{ji,t-1}, L_{jt}} q_{jt} \left(Y_{jt} - \sum_{i=1}^{N} (1 + r_{jit}) S_{ji,t-1} - w_{jt} L_{jt} \right),$$

subject to equation (2). The FOCs with respect to labor and intermediate goods are given, respectively, by

$$w_{jt} = b_j \frac{Y_{jt}}{L_{jt}}. \tag{A.6}$$

$$1 + r_{jit} = a_{ji} \frac{Y_{jt}}{S_{ji,t-1}}, \tag{A.7}$$

Since production is constant returns to scale, i.e., $b_j + \sum_{i=1}^{N} a_{ji} = 1$, Substituting equation (A.6) and (A.7) respectively into (A.4) and (A.5) yields

$$w_{jt} = \frac{C_{jt}}{\varphi_j}, \tag{A.8}$$

$$\frac{\varphi_i}{C_{it}} = \beta \frac{\varphi_i}{C_{j,t+1}} a_{ji} \frac{Y_{j,t+1}}{S_{jit}}. \tag{A.9}$$

Following Long and Plosser (1983), we will guess and verify that there exists a constant vector $\{\gamma_i\}_{i\in\mathbf{N}}$ such that consumption demand for good i is proportional to output in sector i:

$$C_{it} = \frac{\varphi_i}{\gamma_i} Y_{it}. \tag{A.10}$$

Then

$$\frac{\varphi_j}{C_{jt}} = q_{jt}\mu_t \text{ for } j \in \mathbf{N}, \tag{A.11}$$

Since $\sum_{j=1}^{N} \varphi_j = 1$, summation of equation (A.2) over j yields

$$\mu_t = \frac{1}{\sum_{j=1}^{N} q_{jt} C_{jt}} = \frac{1}{C_t}. \tag{A.12}$$

Then combining equation (A.2) and (A.10) yields

$$\gamma_j = q_{jt} Y_{jt} \mu_t = \frac{q_{jt} Y_{jt}}{C_t}. \tag{A.13}$$

Moreover, substituting equation (A.10) into (A.8) and (A.9) respectively yields

$$L_{jt} = \gamma_j b_j, \tag{A.14}$$

and

$$S_{jit} = \beta \frac{\gamma_j}{\gamma_i} a_{ji} Y_{jt}, \tag{A.15}$$

Note that equation (A.14) is different from the policy function on labor in Long and Plosser (1983). As shown below in Remark 2, once we are back to the preference by Long and Plosser, then all the allocation under the current decentralized economy will coincide with that in Long and Plosser (1983).

It remains to pin down the constant coefficient vector $\{\gamma_i\}_{i\in\mathbf{N}}$. First, the clearing condition in goods j is given by

$$C_{jt} + \sum_{i=1}^{N} S_{ijt} = Y_{jt}. \tag{A.16}$$

Substituting equation (A.10) and (A.15) into (A.16) yields

$$\gamma_j = \varphi_j + \beta \sum_{i=1}^{N} a_{ij}\gamma_i, \text{ for } j \in \mathbf{N}, \tag{A.17}$$

which can be rewritten in vector form as $\boldsymbol{\gamma}' = \boldsymbol{\varphi}' + \beta\boldsymbol{\gamma}'\mathbf{A}$. Then $\boldsymbol{\gamma}'$ is obtained as

$$\boldsymbol{\gamma}' = \boldsymbol{\varphi}' \left(I - \beta\mathbf{A}\right)^{-1}, \tag{A.18}$$

where $\boldsymbol{\gamma}'$ and $\boldsymbol{\varphi}'$ denote, respectively, the $1 \times N$ vector of $\{\gamma_i\}$ and the $1 \times N$ vector of $\{\varphi_i\}$, and $\mathbf{A} = (a_{ij})_{N\times N}$ denotes the $N \times N$ matrix of input-output elasticity coefficients.

Remark 2 *In the original setup of Long and Plosser (1983), the preference of the representative household is given by*

$$u(C_t, Z_t) = \varphi_0 \ln Z_t + \sum_{i=1}^{N} \varphi_i \ln C_{it}.$$

Then the FOC on leisure Z_t can be obtained as

$$\frac{\varphi_0}{Z_t} = \frac{\varphi_i}{C_{it}} b_i \frac{Y_{it}}{L_{it}}, \tag{A.19}$$

Substituting equation (A.10) into (A.19) yields

$$L_{it} = \frac{\gamma_i b_i}{\varphi_0} Z_t, \tag{A.20}$$

In turn, substituting equation (A.20) into (3) yields

$$Z_t = \frac{\varphi_0}{\varphi_0 + \sum_{i=1}^{N} \gamma_i b_i} H, \tag{A.21}$$

Finally, by combining (3) and (A.21), we obtain labor supply in sector i as

$$L_{it} = \frac{\gamma_i b_i}{\varphi_0 + \sum_{j=1}^{N} \gamma_j b_j} H. \tag{A.22}$$

Note that equation (A.21) and (A.22) delivers the same allocation on leisure and labor supply in Long and Plosser (1983) using social-planner approach. Also the same allocation on consumption C_{it} and S_{ijt} as obtained in equation (A.10) and (A.15).

Proof of Proposition 2: Denote $\{\mu_t, \nu_t\}$ as the Lagrangian multipliers for constraint (16) and (13) respectively, and assume that $\mathbf{l}_{jt}$ adopts interior solutions. Then the Lagrangian is given by

$$\mathcal{L} = \theta_t \cdot \left(\sum_{j=1}^{N} \varphi_j \ln c_{jt} \right) + \beta \mathbb{E}_t V_{t+1} \left(\frac{m_{t+1}}{P_{t+1}} \right) + \mu_t \left(x_t - \frac{m_{t+1}}{P_t} - \sum_{j=1}^{N} q_{jt} c_{jt} \right) + \nu_t \frac{m_{t+1}}{P_t}.$$

Accordingly, the FOC on $\{\mathbf{c}_t, m_{t+1}, s_{jit,} l_{jt}\}$ yields

$$\theta_t \frac{\varphi_j}{c_{jt}} = q_{jt} \mu_t, \text{ for all } j \in \mathbf{N}, \tag{A.23}$$

$$\mu_t = \beta \mathbb{E}_t \frac{\partial V_{t+1}}{\partial \tilde{m}_{t+1}} \frac{P_t}{P_{t+1}} + v_t, \tag{A.24}$$

$$q_{it} \mu_t = \beta \mathbb{E}_t (1 + r_{ji,t+1}) q_{j,t+1} \mu_{t+1}, \tag{A.25}$$

$$\frac{1}{q_{jt}w_{jt}} = \frac{1}{w_t} = \left(\int \frac{\partial J_t}{\partial x_t} d\mathbf{F}\right) \text{ for } j \in \mathbf{N}, \tag{A.26}$$

where $\tilde{m}_t \equiv \frac{m_t}{P_t}$ denotes real money balance, and as a recap, we have denoted in the main context that $\mathbf{N} = \{1, ..., N\}$. Thus the law of one price implies $w_t \equiv q_{jt}w_{jt}$, for all $j \in \mathbf{N}$.

Second, the envelope theorem implies

$$\frac{\partial J_t}{\partial x_t} = \mu_t, \tag{A.27}$$

$$\frac{\partial V_t}{\partial \tilde{m}_t} = \int \frac{\partial J_t}{\partial x_t} d\mathbf{F}. \tag{A.28}$$

Then the FOCs can be further formulated as

$$\theta_t \frac{\varphi_j}{c_{jt}} = q_{jt}\mu_t, \text{ where } j \in \mathbf{N}, \tag{A.29}$$

$$\mu_t = \beta \mathbb{E}_t \mu_{t+1} \frac{P_t}{P_{t+1}} + v_t, \tag{A.30}$$

$$\frac{q_{it}}{w_t} = \beta \mathbb{E}_t \left(1 + r_{ji,t+1}\right) \frac{q_{j,t+1}}{w_{t+1}}. \tag{A.31}$$

$$\mu_t = \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \frac{1}{w_{t+1}} + v_t. \tag{A.32}$$

As proved below, it turns out that the decision rules for consumption and money demand are characterized by a cutoff strategy. Denote the cutoff value as θ_t^*, which is endogenous, and will be characterized as well.

Case A: $\theta_t \leq \theta_t^*$. In this case, $m_{t+1} \geq 0$, $v_t = 0$, and thus

$$\mu_t = \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \frac{1}{w_{t+1}}, \tag{A.33}$$

and then

$$c_{jt} = \frac{\varphi_j}{q_{jt}} \frac{\theta_t}{\mu_t}. \tag{A.34}$$

In turn, the budget constraint implies that

$$\frac{m_{t+1}}{P_t} = x_t - \sum_{j=1}^{N} c_{jt} = x_t - \frac{\theta_t}{\mu_t} \geq 0, \tag{A.35}$$

and thus

$$\theta_t \leq \theta_t^* \equiv \mu_t x_t. \tag{A.36}$$

Then $\mu_t = \frac{\theta_t^*}{x_t}$, and we have

$$\theta_t \frac{\varphi_j}{c_{jt}} = q_{jt}\mu_t = q_{jt}\frac{\theta_t^*}{x_t}, \tag{A.37}$$

or, equivalently,

$$c_{jt} = \frac{\theta_t}{\theta_t^*} \frac{\varphi_j}{q_{jt}} x_t. \tag{A.38}$$

Besides,

$$\frac{\theta_t^*}{x_t} = \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \frac{1}{w_{t+1}}. \tag{A.39}$$

Moreover, using $\mu_t = \frac{\theta_t^*}{x_t}$, the budget constraint implies

$$\frac{m_{t+1}}{P_t} = \frac{\theta_t^* - \theta_t}{\theta_t^*} x_t. \tag{A.40}$$

Case B: when $\theta_t > \theta_t^*$. In this case, $m_{t+1} = 0$. Then $\sum_{j=1}^{N} q_{jt} c_{jt} = x_t$. In turn,

$$c_{jt} = \frac{\varphi_j}{q_{jt}} x_t, \text{ where } j \in \mathbf{N}.$$

In sum, for any $\theta_t \in (\theta_{\min}, \theta_{\max})$, the multiplier μ_t is determined by

$$\mu_t = \frac{\max\{\theta_t^*, \theta_t\}}{x_t}, \tag{A.41}$$

Correspondingly, individual consumption on good j is given by

$$c_{jt} = \frac{\varphi_j}{q_{jt}} \min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} x_t, \text{ where } j \in \mathbf{N}, \tag{A.42}$$

and the individual real money holding rebalanced by

$$\frac{m_{t+1}}{P_t} = \max\left\{\frac{\theta_t^* - \theta_t}{\theta_t^*}, 0\right\} x_t. \tag{A.43}$$

Moreover, the cash in hand can be characterized as

$$x_t = w_t \theta_t^* R(\theta_t^*). \tag{A.44}$$

Proof of Proposition 3: Integrating equation (20) yields

$$C_{jt} = \frac{\varphi_j}{q_{jt}} D(\theta_t^*) X_t, \tag{A.45}$$

where $D(\theta_t^*) \equiv \int_{\theta_{\min}}^{\theta_{\max}} \min\left(1, \frac{\theta_t}{\theta_t^*}\right) d\mathbf{F}$. In the same spirit, we know that

$$X_t = w_t \theta_t^* R(\theta_t^*).$$

Integrating over equation (15) yields

$$\begin{aligned}X_t &= \frac{M_t+\tau_t}{P_t}+\sum_{j=1}^{N}q_{jt}\left[\sum_{i=1}^{N}\left(\left(1+r_{jit}\right)S_{ji,t-1}-S_{ijt}\right)\right]+\sum_{j=1}^{N}q_{jt}w_jL_{jt}\\ &= \frac{M_t+\tau_t}{P_t}+\sum_{j=1}^{N}q_{jt}\left[Y_{jt}+(1-\delta)\sum_{i=1}^{N}S_{ji,t-1}-\sum_{i=1}^{N}S_{ijt}\right]\\ &= \frac{M_t+\tau_t}{P_t}+\sum_{j=1}^{N}q_{jt}C_{jt}\\ &= \frac{M_t+\tau_t}{P_t}+C_t,\end{aligned}$$

where the second equality uses the results on factor prices in equation (28) and (29), the third equality uses the budget constraint, i.e.,

$$C_{jt}+\sum_{i=1}^{N}S_{ijt}=Y_{jt}+(1-\delta)\sum_{i=1}^{N}S_{ji,t-1},$$

and the last equality uses the definition of C_t, and C_{jt} in equation (A.45), such that

$$C_t=D\left(\theta_t^*\right)X_t,$$

In turn, using the clearing condition in the money market, i.e., $M_{t+1}=M_t+\tau_t$, we know that

$$\frac{M_{t+1}}{P_t}=\frac{M_t+\tau_t}{P_t}=X_t-C_t-\sum_{j=1}^{N}q_{jt}G_{jt}=H\left(\theta_t^*\right)X_t, \tag{A.46}$$

where $H\left(\theta_t^*\right)\equiv 1-D\left(\theta_t^*\right)$.

Proof of Proposition 4: Motivated by equation (A.10), i.e., the policy function on consumption in Long and Plosser (1983), we conjecture that there exists γ_{jt} such that

$$C_{jt}=\frac{\varphi_j}{\gamma_{jt}}Y_{jt},\ \text{ for } j\in\mathbf{N}. \tag{A.47}$$

Substituting equation (30) into (A.47) yields

$$\gamma_{jt}=\frac{q_{jt}Y_{jt}}{C_t}. \tag{A.48}$$

Note that equation (A.48) can be rewritten as

$$q_{jt}=\frac{\gamma_{jt}C_t}{Y_{jt}}. \tag{A.49}$$

Consequently, when $\delta=1$, substituting (37) into (35) yields

$$S_{ijt}=\left(\mathbb{E}_t\frac{\gamma_{i,t+1}}{\gamma_{it}}\frac{w_t}{w_{t+1}}\frac{C_{t+1}}{C_t}\right)\left(\frac{\beta\gamma_{it}a_{ij}Y_{jt}}{\gamma_{jt}}\right). \tag{A.50}$$

Note that

$$\frac{\gamma_{i,t+1}}{\gamma_{it}}\frac{w_t}{w_{t+1}}\frac{C_{t+1}}{C_t} = \frac{\frac{q_{i,t+1}Y_{i,t+1}}{w_{t+1}}}{\frac{q_{it}Y_{it}}{w_t}} = \frac{\frac{Y_{i,t+1}}{w_{i,t+1}}}{\frac{Y_{jt}}{w_{jt}}} = \frac{L_{i,t+1}}{L_{i,t}}. \tag{A.51}$$

where the first, second, and last equality holds because of equation (A.48), (38), and (36) respectively.

Combining equation (A.51) and (A.50) yields that

$$S_{ijt} = \frac{\beta\gamma_{it}a_{ijt}}{\gamma_{jt}}Y_{jt}, \text{ for } i,j \in \mathbf{N}. \tag{A.52}$$

where

$$a_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t\left(\frac{L_{i,t+1}}{L_{i,t}}\right).$$

Furthermore, given $\delta = 1$, as the restriction made in Long and Plosser (1983), the resource constraint in equation (39) can be simplified as

$$C_{jt} + \sum_{i=1}^{N} S_{ijt} = Y_{jt}, \text{ for } j \in \mathbf{N}, \tag{A.53}$$

Substituting (A.47) and (A.52) into (A.53) then yields that

$$\gamma_{jt} = \varphi_j + \sum_{i=1}^{N} \beta\gamma_{it}a_{ijt},$$

which can be written as a more compact way as

$$\boldsymbol{\gamma}_t' = \boldsymbol{\varphi}'\left(I - \beta\mathbf{A}_t\right)^{-1}.$$

where $\boldsymbol{\gamma}_t'$ and $\boldsymbol{\varphi}'$ denote $1 \times N$ vector of $\{\gamma_{it}\}$ and $\{\varphi_i\}$ respectively, and $\mathbf{A}_t = (a_{ijt})_{N\times N}$ the adjusted $N \times N$ I-O table, with $a_{ijt} \equiv a_{ij} \cdot \mathbb{E}_t\left(L_{i,t+1}/L_{i,t}\right)$.

Proof of Proposition 5: Rewriting the FOC on L_{jt}, i.e., $L_{jt} = b_j\frac{Y_{jt}}{w_{jt}}$, yields

$$\frac{L_{jt}}{b_j} = \frac{Y_{jt}}{w_{jt}} = \frac{q_{jt}Y_{jt}}{q_{jt}w_{jt}} = \frac{\gamma_{jt}C_t}{w_t} = \gamma_{jt}D_tR_t\theta_t^*.$$

Then γ_{jt} is obtained as

$$\gamma_{jt} = \frac{L_{jt}}{b_j}\frac{1}{D_tR_t\theta_t^*},$$

which can be rewritten in a compact way as

$$\gamma_t' = \tilde{L}_t'\frac{1}{D_tR_t\theta_t^*}.$$

Substituting equation (51) into the above equation gives

$$\tilde{L}_t' \frac{1}{D_t R_t \theta_t^*} = \varphi' \left(I - \beta A_t\right)^{-1}.$$

where $\tilde{\mathbf{L}}_t$ is a $N \times 1$ vector with a typical element as $\tilde{L}_{jt} \equiv \frac{L_{jt}}{b_{jt}}$. Consequently, by denoting $Z_t \equiv D_t\left(\theta_t^*\right) R_t\left(\theta_t^*\right)\theta_t^*$, we obtain

$$\tilde{\mathbf{L}}_t' = \tilde{\mathbf{L}}_{t+1}' \beta A + Z_t \varphi'.$$

Proof of the Analysis of Steady State: Equation (24) implies that $R\left(\theta^*\right)$ strictly decreases with θ^* over $\left(\theta_{\min}, \theta_{\max}\right)$ with the boundary limit as

$$\lim_{\theta^* \to \theta_{\min}} R\left(\theta^*\right) = \frac{\mathbb{E}\left(\theta\right)}{\theta_{\min}}, \lim_{\theta^* \to \theta_{\max}} R\left(\theta^*\right) = 1.$$

Consequently, there exists a (unique) solution to equation (54) on θ^* if and only if

$$1 \leq \frac{1+\pi}{\beta} \leq \frac{\mathbb{E}\left(\theta\right)}{\theta_{\min}},$$

or equivalently,

$$\beta - 1 \leq \pi \leq \beta \frac{\mathbb{E}\left(\theta\right)}{\theta_{\min}} - 1.$$

If Pareto, with $F\left(\theta\right) = 1 - \left(\theta/\theta_{\min}\right)^{-\eta}$. Then $\mathbb{E}\left(\theta\right) = \frac{\eta}{\eta-1}\theta_{\min} = 1$, and thus $\theta_{\min} = \frac{\eta-1}{\eta}$, and $\mathbb{E}\left(\theta | \theta \geq \theta^*\right) = \frac{\eta}{\eta-1}\theta^* = \frac{\theta^*}{\theta_{\min}}$.

$$\begin{aligned} R\left(\theta^*\right) &\equiv \int_{\theta_{\min}}^{\theta_{\max}} \max\left(1, \frac{\theta}{\theta^*}\right) d\mathbf{F} \\ &= \int_{\theta_{\min}}^{\theta^*} d\mathbf{F} + \int_{\theta^*}^{\theta_{\max}} \frac{\theta}{\theta^*} d\mathbf{F} \\ &= F\left(\theta^*\right) + \left(1 - F\left(\theta^*\right)\right) \mathbb{E}\left(\frac{\theta}{\theta^*} | \theta \geq \theta^*\right) \\ &= 1 + \frac{\left(\theta^*/\theta_{\min}\right)^{-\eta}}{\eta - 1}, \end{aligned}$$

and then we can immediately obtain the analytical solution to θ^*:

$$\theta^*/\theta_{\min} = \left(\left(R-1\right)\left(\eta-1\right)\right)^{-\frac{1}{\eta}} = \left(\left(\frac{1+\pi}{\beta} - 1\right)\left(\eta - 1\right)\right)^{-\frac{1}{\eta}},$$

as shown in equation (54). Meanwhile, the proportion of household being constrained, i.e., the probability that $\theta \geq \theta^*$ is given by

$$1 - F\left(\theta^*\right) = \left(\frac{1+\pi}{\beta} - 1\right)\left(\eta - 1\right).$$

In steady state, equation (35) and (37) respectively implies that

$$1 + r_{ij} = \frac{q_j/q_i}{\beta},$$

and

$$r_{ij} + \delta = a_{ij}\frac{Y_i}{S_{ij}}.$$

Combining those two equations yields

$$\frac{S_{ij}}{Y_i} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)}.$$

Since $b_i + \sum_{j=1}^{N} a_{ij} = 1$, we can rewrite equation (27), i.e., the production technology in any sector i as

$$Y_i^{b_i} \prod_{j=1}^{N} Y_i^{a_{ij}} = \lambda_i L_i^{b_i} \prod_{j=1}^{N} S_{ij}^{a_{ij}} = Y_i,$$

which can be further simplified as

$$\left(\frac{Y_i}{L_i}\right)^{b_i} = \lambda_i \prod_{j=1}^{N} \left(\frac{S_{ij}}{Y_i}\right)^{a_{ij}}. \tag{A.54}$$

Substituting equation (A.54) into (36) yields the wage rate in sector i as

$$w_i = b_i \frac{Y_i}{L_i} = b_i \left[\lambda_i \prod_{j=1}^{N} \left(\frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)}\right)^{a_{ij}}\right]^{\frac{1}{b_i}}. \tag{A.55}$$

Note that, given w, we have N equations on N variables $(q_1, ..., q_N)$ such that

$$w = b_i \left[\lambda_i \prod_{j=1}^{N} \left(\frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)}\right)^{a_{ij}}\right]^{\frac{1}{b_i}} q_i \equiv \Gamma_i(q_1, ..., q_N) \text{ for } i \in \mathbf{N}. \tag{A.56}$$

Therefore $q = q(w)$. The normalization of price index is given by

$$\prod_{j=1}^{N} \left(\frac{q_j}{\varphi_j}\right)^{\varphi_j} = 1.$$

By both sides by log yields

$$\sum_{j=1}^{N} \varphi_j \ln q_j = \sum_{j=1}^{N} \varphi_j \ln \varphi_j,$$

or, equivalently,

$$\boldsymbol{\varphi}'\widetilde{\mathbf{q}} = \boldsymbol{\varphi}' \ln \boldsymbol{\varphi}, \tag{A.57}$$

where $\boldsymbol{\varphi} = [\varphi_1, ..., \varphi_N]$. Then substituting $q = q(w)$ into (A.57) yields w.

Furthermore, if $\delta = 1$, given $\mathbf{q} = (q_1, ..., q_N)'$ is determined by the simultaneous equation system, $q_i \omega_i(\mathbf{q}) = w$ for $i \in N$, or, equivalently,

$$Diag(\boldsymbol{\omega}(\mathbf{q})) \cdot \mathbf{q} = \mathbf{w}. \tag{A.58}$$

where for sector $i \in N$

We can an analytical solution for $\mathbf{q}$ as below,

$$\widetilde{\mathbf{q}} = (\mathbf{I} - \mathbf{A})^{-1} \cdot (\tilde{w}\mathbf{b} - \mathbf{d}), \tag{A.59}$$

where $\widetilde{\mathbf{q}} = \ln \mathbf{q} = [\ln q_1, ..., \ln q_N]$, and $\mathbf{d}$ is $N \times 1$ with a typical element of d is

$$d_i(w) \equiv b_i \ln w - \left(b_i \ln b_i + \ln \lambda_i + \sum_{j=1}^{N} a_{ij} (\ln \beta + \ln a_{ij}) \right)$$

The wage rate w is obtained by using the normalization on q, i.e.,

$$\prod_{j=1}^{N} \left(\frac{q_j}{\varphi_j} \right)^{\varphi_j} = 1. \tag{A.60}$$

with

$$w_i \equiv b_i \left[\lambda_i \prod_{j=1}^{N} \left(\frac{\beta a_{ij}}{q_j / q_i} \right)^{a_{ij}} \right]^{\frac{1}{b_i}}, \tag{A.61}$$

Denote $\tilde{w} = \ln w$. Then

$$\begin{aligned} d &= \tilde{w} \cdot b - \left(Diag(b)\, \tilde{b} + \tilde{\lambda} + A\tilde{\beta} I_{N\times 1} + e \right) \\ &= \mathbf{b} \ln(w) - \mathbf{d}, \end{aligned}$$

where $e_i = A(i,:) \cdot \ln A(i,:)$. Then we have

$$\boldsymbol{\varphi}' (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{d} = \boldsymbol{\varphi}' \ln \boldsymbol{\varphi},$$

and thus

$$\begin{aligned} \tilde{w} &= \frac{\boldsymbol{\varphi}' \left[(\mathbf{I} - \mathbf{A})^{\mathbf{-1}} \left(\mathbf{Diag}(\mathbf{b})\, \tilde{\mathbf{b}} + \tilde{\boldsymbol{\lambda}} + \mathbf{A} \tilde{\boldsymbol{\beta}} \mathbf{I}_{\mathbf{N}\times\mathbf{1}} + \mathbf{e} \right) + \ln \boldsymbol{\varphi} \right]}{\boldsymbol{\varphi}' (\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}} \\ &= \frac{\boldsymbol{\varphi}' \left[(\mathbf{I} - \mathbf{A})^{\mathbf{-1}} \mathbf{dd} + \ln \boldsymbol{\varphi} \right]}{\boldsymbol{\varphi}' (\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}}. \end{aligned}$$

Then

$$w = \exp(\ln w) = \exp(\tilde{w}).$$

where $\boldsymbol{\varphi} = [\varphi_1, ..., \varphi_N]'$.

After solving w, we immediately obtain $(q_i, w_i)_{i\in\mathbf{N}}$, and then

$$\begin{aligned} X &= w\theta^* R(\theta^*), \\ C_j &= \frac{\varphi_j}{q_j} D(\theta^*) X, \\ \frac{M_{t+1}}{P_t} &= H(\theta^*) X \text{ for all } t. \end{aligned}$$

where $H(\theta^*) = 1 - D(\theta^*)$.

Moreover, since $L_{i,t+1} = L_{it}$ in steady state, a_{ijt} coincides with a_{ij}. Then equation (49) and (50) immediately imply that

$$Y_j = \frac{\gamma_j}{\varphi_j} C_j,$$

$$S_{ij} = \frac{\beta\gamma_i a_{ij}}{\gamma_j} Y_j,$$

where

$$\boldsymbol{\gamma}' = \boldsymbol{\varphi}' (I - \beta\mathbf{A})^{-1}.$$

In turn, capital and labor demand is obtained

$$S_{ij} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} Y_i, \text{ for } i, j \in \mathbf{N},$$

and

$$L_i = \frac{b_i}{w_i} Y_i, \text{ for } i \in \mathbf{N}.$$

When $G_j > 0$, then equation (A.53) is generalized as

$$C_j + \sum_{i=1}^{N} S_{ij} + G_j = Y_j. \tag{A.62}$$

Assume

$$g = \frac{G_j}{Y_j}.$$

Then substituting

$$C_j = \frac{\varphi_j}{\gamma_j} Y_j,$$

and

$$S_{ij} = \frac{\beta\gamma_i a_{ij}}{\gamma_j} Y_j,$$

into (A.62) yields

$$(1-g)\gamma_j = \varphi_j + \beta \sum_{i=1}^{N} a_{ij}\gamma_i, \tag{A.63}$$

and thus

$$(1-g)\gamma = \varphi + \beta A\gamma,$$

and thus

$$\gamma = ((1-g)\, I - \beta A)^{-1}\, \varphi.$$

If $g = 0$, then it is reduced to the baseline case.

Now we address the endogenous labor wedge Z. Note that the labor wedge is immediately obtained from equation (52) of Prop 5.

Moreover, we know that, in steady state,

$$C_j = \frac{\varphi_j}{\gamma_j} Y_j, \text{ for } j \in \mathbf{N},$$

$$S_{ij} = \frac{\beta \gamma_i a_{ij}}{\gamma_j} Y_j, \text{ for } i, j \in \mathbf{N},$$

$$L_j = D(\theta^*)\, R(\theta^*)\, \theta^* b_j \gamma_j = Z(\theta^*)\, b_j \gamma_j$$

where $\boldsymbol{\gamma}' = \boldsymbol{\varphi}' (I - \beta \mathbf{A})^{-1}$ and $R(\theta^*) = \frac{1+\pi}{\beta}$.

Since

$$Y_i = \lambda_i L_i^{b_i} \prod_{j=1}^{N} S_{ij}^{a_{ij}},$$

taking log yields

$$\begin{aligned}
\ln Y_i &= \ln \lambda_i + b_i \ln L_i + \sum_{j=1}^{N} a_{ij} \ln S_{ij} \\
&= \ln \lambda_i + b_i \ln (Z(\theta^*)\, b_i \gamma_i) + \sum_{j=1}^{N} a_{ij} \ln \left(\frac{\beta \gamma_i a_{ij}}{\gamma_j} Y_j \right) \\
&= \ln \lambda_i + b_i \ln Z(\theta^*) + b_i \ln (b_i \gamma_i) + \sum_{j=1}^{N} a_{ij} \ln \left(\frac{\beta \gamma_i a_{ij}}{\gamma_j} \right) + \sum_{j=1}^{N} a_{ij} \ln Y_j \\
&= \omega_i + b_i \ln Z + \sum_{j=1}^{N} a_{ij} \ln Y_j
\end{aligned}$$

where

$$\begin{aligned}
\omega_i &= \ln \lambda_i + b_i \ln (b_i \gamma_i) + \sum_{j=1}^{N} a_{ij} \ln \left(\frac{\beta \gamma_i a_{ij}}{\gamma_j} \right) \\
&= \ln \lambda_i + \ln \gamma_i + b_i \ln b_i + (1 - b_i) \ln \beta + \sum_{j=1}^{N} a_{ij} \ln \frac{a_{ij}}{\gamma_j}
\end{aligned}$$

Then

$$y_i = \omega_i + b_i \ln Z + \sum_{j=1}^{N} a_{ij} y_j,$$

and thus

$$y = \omega + (\ln Z)\, b + Ay.$$

Then

$$\begin{aligned} y &= (I-A)^{-1}(\omega + (\ln Z)\, b) \\ &= (I-A)^{-1}\omega + (\ln Z)(I-A)^{-1} b \\ &= y^{LP} + (\ln Z)(I-A)^{-1} b \end{aligned}$$

Thus we obtain the distributional effect of monetary policy on sectoral TFP.

$$\ln Y - \ln Y^{LP} = (\ln Z) \cdot (I-A)^{-1} b.$$

where $\ln Y = (\ln Y_1, ..., \ln Y_N)'$.

B Dynamical System

We can obtain the dynamical system of equations that govern the path of $\{\mathbf{q}_t, \mathbf{r}_t, \mathbf{w}_{it}, \mathbf{L}_{it}, \mathbf{C}_t, \mathbf{S}_t, \mathbf{Y}_t, X_t, \theta_t^*, M_{t+1},$
in a competitive equilibrium.

$$C_{jt} = \frac{\varphi_{jt}}{q_{jt}} D(\theta_t^*) X_t, \text{ for } j \in \mathbf{N}, \tag{B.1}$$

$$\frac{M_{t+1}}{P_t} = H(\theta_t^*) X_t, \tag{B.2}$$

$$X_t = w_t \theta_t^* R(\theta_t^*), \tag{B.3}$$

$$\frac{1}{w_t} = \beta \mathbb{E}_t \frac{P_t}{P_{t+1}} \frac{1}{w_{t+1}} R(\theta_t^*), \tag{B.4}$$

$$Y_{it} = \lambda_{it} L_{it}^{b_i} \prod_{j=1}^{N} S_{ij,t-1}^{a_{ij}}, \text{ for } i \in \mathbf{N}, \tag{B.5}$$

$$\frac{q_{jt}}{w_t} = \beta \mathbb{E}_t (1 + r_{ij,t+1}) \frac{q_{i,t+1}}{w_{t+1}}, \text{ for } i, j \in \mathbf{N}, \tag{B.6}$$

$$w_{jt} = b_j \frac{Y_{jt}}{L_{jt}}, \text{ for } j \in \mathbf{N}, \tag{B.7}$$

$$\begin{aligned} r_{ijt} + \delta &= a_{ij} \frac{Y_{it}}{S_{ij,t-1}}, \text{ for } i, j \in \mathbf{N}, && \text{(B.8)} \\ R_{ij,t} &= 1 + r_{ij,t+1} = a_{ij} \frac{Y_{i,t+1}}{S_{ijt}}, \text{ for } i, j \in \mathbf{N} \end{aligned}$$

$$w_t = q_{jt} w_{jt}, \text{ for } j \in \mathbf{N}, \tag{B.9}$$

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} S_{ij,t+1} + G_{jt} - (1-\delta) \sum_{i=1}^{N} S_{jit}, \text{ for } j \in \mathbf{N}, \tag{B.10}$$

or, since $\delta = 1$, we have

$$Y_{jt} = C_{jt} + G_{jt} + \sum_{i=1}^{N} S_{ij,t+1}.$$

where $D(\theta_t^*) \equiv \int_{\theta_{\min}}^{\theta_{\max}} \min\left(1, \frac{\theta_t}{\theta_t^*}\right) d\mathbf{F}$, $H(\theta_t^*) \equiv 1 - D(\theta_t^*) = \int_{\theta_{\min}}^{\theta_{\max}} \max\left(0, \frac{\theta_t^* - \theta_t}{\theta_t^*}\right) d\mathbf{F}$, and $R(\theta_t^*) \equiv \int_{\theta_{\min}}^{\theta_{\max}} \max\left(1, \frac{\theta_t}{\theta_t^*}\right) d\mathbf{F}$.

Log Linearization

$$\hat{C}_{jt} = \hat{D}_t + \hat{X}_t - \hat{q}_{jt}, \text{ for } j \in \mathbf{N}, \tag{B.11}$$

$$\hat{M}_{t+1} - \hat{P}_t = \hat{H}_t + \hat{X}_t, \tag{B.12}$$

$$\hat{X}_t = \hat{w}_t + \hat{\theta}_t^* + \hat{R}_t, \tag{B.13}$$

$$\hat{w}_t = \hat{P}_{t+1} - \hat{P}_t + \hat{w}_{t+1} - \hat{R}_t, \tag{B.14}$$

$$\hat{q}_{jt} - \hat{w}_t = \hat{Y}_{i,t+1} - \hat{S}_{ijt} + \hat{q}_{i,t+1} - \hat{w}_{t+1}, \text{ for } i, j \in \mathbf{N}, \tag{B.15}$$

$$\hat{w}_{jt} = \hat{Y}_{jt} - \hat{L}_{jt}, \text{ for } j \in \mathbf{N}, \tag{B.16}$$

$$\hat{w}_t = \hat{q}_{jt} + \hat{w}_{jt}, \text{ for } j \in \mathbf{N}, \tag{B.17}$$

$$\hat{Y}_{jt} = \frac{C_j}{Y_j}\hat{C}_{jt} + \sum_{i=1}^{N} \frac{S_{ij}}{Y_j}\hat{S}_{ijt}, \text{ for } j \in \mathbf{N}, \tag{B.18}$$

$$\hat{Y}_{it} = \hat{\lambda}_{it} + b_i\hat{L}_{it} + \sum_{j=1}^{N} a_{ij}\hat{S}_{ij,t-1}, \tag{B.19}$$

$$\hat{M}_{t+1} = \rho_m \hat{M}_t + \varepsilon_t^m, \tag{B.20}$$

$$\hat{\lambda}_t = \rho_\lambda \hat{\lambda}_t + \varepsilon_t^\lambda, \tag{B.21}$$

where

$$\hat{R}_t = -\eta\left(\frac{R-1}{R}\right)\hat{\theta}_t^*, \tag{B.22}$$

$$\hat{D}_t = -\frac{1}{D}\left(\frac{1}{\theta^*}\hat{\theta}_t^* + R\hat{R}_t\right), \tag{B.23}$$

$$H_t = -\frac{D}{H}\hat{D}_t, \tag{B.24}$$

$$\hat{v}_t = \hat{D}_t - \hat{H}_t, \tag{B.25}$$

$$\hat{C}_t = \hat{D}_t + \hat{X}_t.$$

and

$$R = \frac{1+\pi}{\beta},$$

$$\theta^* = \left[\left(\frac{1+\pi}{\beta} - 1 \right) (\eta - 1) \right]^{-\frac{1}{\eta}} \theta_{\min},$$

$$\theta_{\min} = 1 - \frac{1}{\eta},$$

$$D = 1 + \frac{\mathbb{E}(\theta)}{\theta^*} - R = 1 + \frac{1}{\theta^*} - R,$$

$$H = 1 - D,$$

$$\frac{M}{P} = Hw\theta^* R, \quad \text{real money balance},$$

$$v = \frac{D}{H},$$

with restrictions on π satisfying

$$\beta - 1 \equiv \pi_{\min} \leq \pi \leq \pi_{\max} \equiv \frac{\beta}{\theta_{\min}} - 1.$$

C Steady State When $\delta < 1$

In general, when $\delta < 1$, the Euler equation on S_{ij} implies

$$\frac{S_{ij}}{Y_i} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)}.$$

Since $b_i + \sum_{j=1}^{N} a_{ij} = 1$, we can rewrite the production technology in any sector i as

$$Y_i^{b_i} \prod_{j=1}^{N} Y_i^{a_{ij}} = \lambda_i L_i^{b_i} \prod_{j=1}^{N} S_{ij}^{a_{ij}} = Y_i,$$

which can be further simplified as

$$\left(\frac{Y_i}{L_i} \right)^{b_i} = \lambda_i \prod_{j=1}^{N} \left(\frac{S_{ij}}{Y_i} \right)^{a_{ij}}.$$

and thus the wage rate in sector i is obtained

$$w_i = b_i \frac{Y_i}{L_i} = b_i \left[\lambda_i \prod_{j=1}^{N} \left(\frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} \right)^{a_{ij}} \right]^{\frac{1}{b_i}}.$$

Since $w = w_i q_i$ for all $i \in \mathbf{N}$, then we know that, *given* w, the above equation implies that we have N variables $\{q_1, q_2, ..., q_N\}$ with N equations:

$$w = b_i \left[\lambda_i \prod_{j=1}^{N} \left(\frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} \right)^{a_{ij}} \right]^{\frac{1}{b_i}} q_i \equiv \Gamma_i(q_1, ..., q_N) \text{ for } i \in \mathbf{N}.$$

Then we can obtain $q_i = q_i(w)$, and thus we can pin down w by using the normalization of the price index, i.e.,

$$\prod_{j=1}^{N}\left(\frac{q_j(w)}{\varphi_j}\right)^{\varphi_j} = 1.$$

After solving w, then as in the main context, we can obtain (q_i, w_i) for all i. Moreover,

$$\begin{aligned}
X &= w\theta^* R(\theta^*) \\
C_j &= \frac{\pi_j}{q_j} D(\theta^*) X = \frac{\pi_j}{q_j} w\theta^* R(\theta^*) D(\theta^*) \\
\frac{M_{t+1}}{P_t} &= H(\theta^*) X \text{ for all } t
\end{aligned}$$

where $H(\theta^*) = 1 - D(\theta^*)$.

The resource constraint can be given rewritten as

$$Y_j + (1-\delta)\sum_{i=1}^{N} S_{ji} - \sum_{i=1}^{N} S_{ij} = C_j.$$

where we have proved previously that $S_{ij} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} Y_i$. As a duality, we also have $S_{ji} = \frac{\beta a_{ji}}{q_i/q_j - \beta(1-\delta)} Y_j$. Substituting S_{ij} and S_{ji} into the above equation then yields

$$\left[1 + \sum_{i=1}^{N} \frac{\beta(1-\delta) a_{ji}}{q_i/q_j - \beta(1-\delta)}\right] Y_j - \sum_{i=1}^{N} \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} Y_i = C_j,$$

which can be rewritten as

$$\hat{a}_j Y_j - \sum_{i=1}^{N} \tilde{a}_{ij} Y_i = C_j,$$

where $\hat{a}_j \equiv 1 + \sum_{i=1}^{N} \frac{\beta(1-\delta)a_{ji}}{q_i/q_j - \beta(1-\delta)}$ and $\tilde{a}_{ij} \equiv \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)}$.

Denote $\mathbf{Y}' = [Y_1, ..., Y_N]$ and $\mathbf{C}' = [C_1, ..., C_N]$. Then the above equation can be rewritten as

$$\mathbf{Y}'\widehat{\mathbf{A}} - \mathbf{Y}'\widetilde{\mathbf{A}} = \mathbf{C}',$$

where $\widehat{\mathbf{A}} = Diag(\hat{a}_j)$ and $\widetilde{\mathbf{A}} = (\tilde{a}_{ij})_{N\times N}$. Then

$$\mathbf{Y}' = \mathbf{C}'\left(\widehat{\mathbf{A}} - \widetilde{\mathbf{A}}\right)^{-1}.$$

In turn, capital and labor demand is obtained

$$S_{ij} = \frac{\beta a_{ij}}{q_j/q_i - \beta(1-\delta)} Y_i, \text{ for } i, j \in \mathbf{N},$$

and

$$L_i = \frac{b_i}{w_i} Y_i, \text{ for } i \in \mathbf{N}.$$